An analysis of mixed integer linear sets based on lattice point free convex sets

Kent Andersen
Department of Mathematics, Otto-von-Guericke Universität Magdeburg, Germany
email: andersen@mail.math.uni-magdeburg.de

Quentin Louveaux
Department of Electrical Engineering and Computer Science, University of Liège, Belgium
email: Q.louveaux@ulg.ac.be

Robert Weismantel
Department of Mathematics, Otto-von-Guericke Universität Magdeburg, Germany
email: weismant@mail.math.uni-magdeburg.de

Split cuts are cutting planes for mixed integer programs whose validity is derived from maximal lattice point free polyhedra of the form $S := \{x : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$ called split sets. The set obtained by adding all split cuts is called the split closure, and the split closure is known to be a polyhedron. A split set S has max-facet-width equal to one in the sense that $\max\{\pi^T x : x \in S\} - \min\{\pi^T x : x \in S\} \leq 1$.

In this paper we consider using general lattice point free rational polyhedra to derive valid cuts for mixed integer linear sets. We say that lattice point free polyhedra with max-facet-width equal to w have width size w. A split cut of width size w is then a valid inequality whose validity follows from a lattice point free rational polyhedron of width size w. The w^{th} split closure is the set obtained by adding all valid inequalities of width size at most w.

In general, a relaxation of a mixed integer set can be obtained by adding *any* family of valid inequalities to the linear relaxation. Our main result is a sufficient condition for the addition of a family of rational inequalities to result in a polyhedral relaxation. We then show that a corollary is that the w^{th} split closure is a polyhedron.

Given this result, a natural question is which width size w^* is required to design a finite cutting plane proof for the validity of an inequality. Specifically, for this value w^*, a finite cutting plane proof exists that uses lattice point free rational polyhedra of width size at most w^*, but no finite cutting plane proof that only uses lattice point free rational polyhedra of width size smaller than w^*. We characterize w^* based on the faces of the linear relaxation.

Key words: mixed integer set ; lattice point free convex set ; cutting plane ; split closure

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1. Introduction. We consider a polyhedron in $\mathbb{R}^n$ of the form

$$P := \operatorname{conv}(\{v^i\}_{i\in V}) + \operatorname{cone}(\{r^j\}_{j\in E}), \tag{1}$$

where V and E are finite index sets, $\{v^i\}_{i\in V}$ denotes the vertices of P and $\{r^j\}_{j\in E}$ denotes the extreme rays of P. We assume P is rational, *i.e.*, we assume $\{r^j\}_{j\in E} \subset \mathbb{Z}^n$ and $\{v^i\}_{i\in V} \subset \mathbb{Q}^n$.

We are interested in points in P that have integer values on certain coordinates. For simplicity assume the first $p > 0$ coordinates must have integer values, and let $q := n - p$. The set $N_I := \{1, 2, \ldots, p\}$ is used to index the integer constrained variables and the set $P_I := \{x \in P : x_j \in \mathbb{Z} \text{ for all } j \in N_I\}$ denotes the mixed integer points in P.

The following concepts from convex analysis are needed (see [8] for a presentation of the theory of convex analysis). For a convex set $C \subseteq \mathbb{R}^n$, the interior of C is denoted $\operatorname{int}(C)$, and the relative interior of C is denoted $\operatorname{ri}(C)$ (where $\operatorname{ri}(C) = \operatorname{int}(C)$ when C is full dimensional).

We consider the generalization of *split sets* (see [5]) to lattice point free rational polyhedra (see [7]). A split set is of the form $S^{(\pi,\pi_0)} := \{x \in \mathbb{R}^p : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$, where $(\pi, \pi_0) \in \mathbb{Z}^{p+1}$ and $\pi \neq 0$. Clearly a split set does not have integer points in its interior. In general, a lattice point free convex set is a convex set that does not contain integer points in its relative interior. Lattice point free convex sets that are maximal wrt. inclusion are known to be polyhedra. We call lattice point free rational polyhedra that are maximal wrt. inclusion for *split polyhedra*. A split polyhedron is full dimensional and can be written as the sum of a polytope $\mathcal{P}$ and a linear space $\mathcal{L}$.

A lattice point free convex set is an object that assumes integrality of *all* coordinates. For *mixed* integrality in $\mathbb{R}^{p+q}$, we use a lattice point free convex set $C^x \subset \mathbb{R}^p$ to form a *mixed* integer lattice point free convex set $C \subset \mathbb{R}^n$ of the form $C := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : x \in C^x\}$. A *mixed integer split polyhedron* is then a polyhedron of the form $L := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : x \in L^x\}$, where L^x is a split polyhedron in $\mathbb{R}^p$.

An important measure in this paper of the size of a mixed integer split polyhedron L is the *facet width* of L. The facet width measures how wide a mixed integer split polyhedron is parallel to a given facet. Specifically, given any facet $\pi^T x \geq \pi_0$ of a mixed integer split polyhedron L, the width of L along π is defined to be the number $w(L, \pi) := \max_{x \in L} \pi^T x - \min_{x \in L} \pi^T x$. The *max-facet-width* of a mixed integer split polyhedron L measures how wide L is along any facet of L, *i.e.*, the max-facet-width $w_f(L)$ of L is defined to be the largest of the numbers $w(L, \pi)$ over *all* facet defining inequalities $\pi^T x \geq \pi_0$ for L.

Any mixed integer lattice point free convex set $C \subseteq \mathbb{R}^n$ gives a relaxation of $\text{conv}(P_I)$

$$R(C, P) := \text{conv}(\{x \in P : x \notin \text{ri}(C)\})$$

that satisfies $\text{conv}(P_I) \subseteq R(C, P) \subseteq P$. The set $R(C, P)$ might exclude fractional points in $\text{ri}(C) \cap P$ and give a tighter approximation of $\text{conv}(P_I)$ than P.

Mixed integer split polyhedra L give as tight relaxations of P_I of the form above as possible. Specifically, if $C, C' \subseteq \mathbb{R}^n$ are mixed integer lattice point free convex sets that satisfy $C \subseteq C'$, then $R(C', P) \subseteq R(C, P)$. For a general mixed integer lattice point free convex set C, the set $R(C, P)$ may not be a polyhedron. However, it is sufficient to consider mixed integer split polyhedra, and we show $R(L, P)$ is a polyhedron when L is a mixed integer split polyhedron (Lemma 2.4).

Observe that the set of mixed integer split polyhedra with max-facet-width equal to one are exactly the split sets $S^{(\pi,\pi_0)} = \{x \in \mathbb{R}^n : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$, where $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$, $\pi_j = 0$ for $j > p$ and $\pi \neq 0$. In [5], Cook et. al. considered the set of split sets

$$\mathcal{L}^1 := \{L \subseteq \mathbb{R}^n : L \text{ is a mixed integer split polyhedron satisfying } w_f(L) \leq 1\}$$

and showed that the *split closure*

$$\text{SC}^1 := \cap_{L \in \mathcal{L}^1} R(L, P)$$

is a polyhedron. A natural generalization of the split closure is to allow for mixed integer split polyhedra that have max-facet-width larger than one. For any $w > 0$, define the set of mixed integer split polyhedra

$$\mathcal{L}^w := \{L \subseteq \mathbb{R}^n : L \text{ is a mixed integer split polyhedron satisfying } w_f(L) \leq w\}$$

with max-facet-width at most w. We define the w^{th} *split closure* to be the set

$$\text{SC}^w := \cap_{L \in \mathcal{L}^w} R(L, P).$$

We prove that for any family $\bar{\mathcal{L}} \subseteq \mathcal{L}^w$ of mixed integer split polyhedra with bounded max-facet-width $w > 0$, the set $\cap_{L \in \bar{\mathcal{L}}} R(L, P)$ is a polyhedron (Theorem 4.2). The proof is based on an analysis of cutting planes from an inner representation of the linear relaxation P. In fact, our proof does not use an outer description of P at all. Many of our arguments are obtained by generalizing results of Andersen et. al. [1] from the first split closure to the w^{th} split closure.

Given a family $\{(\delta^l)^T x \geq \delta_0^l\}_{i \in I}$ of rational cutting planes, we provide a sufficient condition for the set $\{x \in P : (\delta^l)^T x \geq \delta_0^l \text{ for all } l \in I\}$ to be a polyhedron (Theorem 3.1). This condition (Assumption 3.1) concerns the number of intersection points between hyperplanes defined from the cuts $\{(\delta^l)^T x \geq \delta_0^l\}_{i \in I}$ and line segments either of the form $\{v^i + \alpha r^j : \alpha \geq 0\}$, or of the form $\{\beta v^i + (1 - \beta)v^k : \beta \in [0, 1]\}$, where $i, k \in V$ denote two vertices of P and $j \in E$ denotes an extreme ray of P. We then show that this condition is satisfied by the collection of facets of the sets $R(L, P)$ for $L \in \bar{\mathcal{L}}$ for any family $\bar{\mathcal{L}} \subseteq \mathcal{L}^w$ of split polyhedra with bounded max-facet-width $w > 0$. It follows that the w^{th} split closure is a polyhedron.

Finite cutting plane proofs for the validity of an inequality for P_I can be designed by using mixed integer split polyhedra. A measure of the complexity of a finite cutting plane proof is the max-facet-width of the mixed integer split polyhedron with the largest max-facet-width in the proof. A measure of the complexity of a valid inequality $\delta^T x \geq \delta_0$ for P_I is the smallest integer $w(\delta, \delta_0)$ for which there exists a finite cutting plane proof of validity of $\delta^T x \geq \delta_0$ for P_I only using mixed integer split polyhedra with max-facet-width at most $w(\delta, \delta_0)$. We give a formula for $w(\delta, \delta_0)$ (Theorem 5.1) that explains geometrically why mixed integer split polyhedra of large width size can be necessary.

The remainder of the paper is organized as follows. In Sect. 2 we present the main results on lattice point free convex sets that are needed in the remainder of the paper. We also present the construction of polyhedral relaxations of P_I from mixed integer split polyhedra. Most results in Sect. 2 can also be found in a paper of Lovász [7]. In Sect. 3 we discuss cutting planes from the viewpoint of an inner representation of P. The main result in Sect. 3 is a sufficient condition for a set obtained by adding an infinite family of cutting planes to be a polyhedron. The structure of the relaxation $R(L, P)$ of P_I obtained from a given mixed integer split polyhedron L is characterized in Sect. 4. The main outcome is that the w^{th} split closure is a polyhedron. Finally, in Sect. 5, we discuss the complexity of finite cutting plane proofs for the validity of an inequality for P_I.

2. Lattice point free convex sets and polyhedral relaxations We now discuss the main object of this paper, namely lattice point free convex sets, which are defined as follows

Definition 2.1 *(Lattice point free convex sets)*
Let $L \subseteq \mathbb{R}^p$ be a convex set. If $\text{ri}(L) \cap \mathbb{Z}^p = \emptyset$, then L is called lattice point free.

The discussion of lattice point free convex sets in this section is based on a paper of Lovász [7]. We are mainly interested in lattice point free convex sets that are maximal wrt. inclusion. Our point of departure is the following characterization of maximal lattice point free convex sets.

Lemma 2.1 *Every maximal lattice point free convex set $L \subseteq \mathbb{R}^p$ is a polyhedron.*

As mentioned in the introduction, we call maximal lattice point free rational polyhedra for *split polyhedra*. Maximal lattice point free polyhedra are not necessarily rational. The polyhedron $C = \{(x_1, x_2) : x_2 = \sqrt{2}x_1, x_1 \geq 0\}$ is an example of a maximal lattice point free set which is *not* a rational polyhedron. However, we will only use maximal lattice point free convex sets to describe (mixed) integer points in rational polyhedra, and for this purpose split polyhedra suffice.

We next argue that the recession cone $0^+(L)$ of a split polyhedron L must be a linear space. This fact follows from the following operation to enlarge any lattice point free convex set $C \subseteq \mathbb{R}^p$. Let $r \in 0^+(C) \cap \mathbb{Q}^p$ be a rational vector in the recession cone of C. We claim that also $C' = C + \text{span}(\{r\})$ is lattice point free. Indeed, if $\bar{x} - \mu r \in \text{ri}(C')$ is integer with $\mu > 0$ and $\bar{x} \in \text{ri}(C)$, then there exists a positive integer $\mu^I > \mu$ such that $\bar{x} - \mu r + \mu^I r = \bar{x} + (\mu^I - \mu)r \in \text{ri}(C) \cap \mathbb{Z}^p$, which contradicts that C is lattice point free. Since the recession cone of a split polyhedron is rational, we therefore have

Lemma 2.2 *Let $L \subseteq \mathbb{R}^p$ be a split polyhedron. Then L can be written in the form $L = \mathcal{P} + \mathcal{L}$, where $\mathcal{P} \subseteq \mathbb{R}^p$ is a rational polytope and $\mathcal{L} \subseteq \mathbb{R}^p$ is a linear space with an integer basis.*

Observe that Lemma 2.2 implies that every split polyhedron $L \subseteq \mathbb{R}^p$ is full dimensional. Indeed, if this was not the case, then we would have $L \subseteq \{x : \mathbb{R}^p : \pi^T x = \pi_0\}$ for some $(\pi, \pi_0) \in \mathbb{Z}^{p+1}$ which implies $L \subseteq \{x : \mathbb{R}^p : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$, and this contradicts that L is maximal and lattice point free.

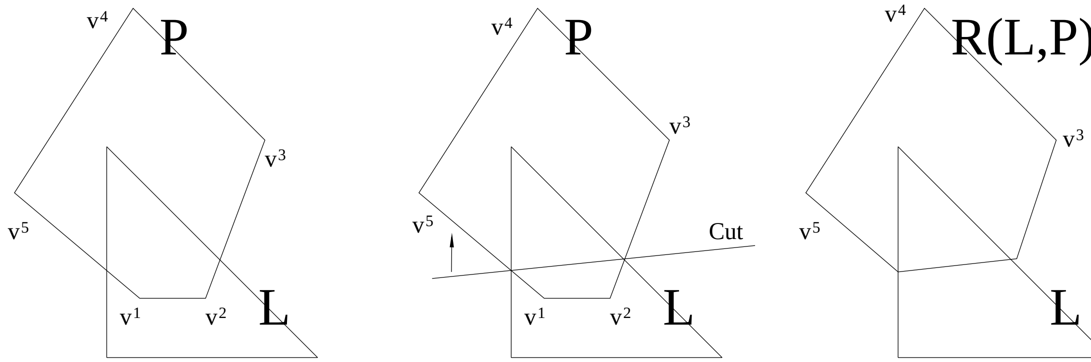
Lemma 2.3 *Every split polyhedron L in $\mathbb{R}^p$ is full dimensional.*

We are interested in using split polyhedra to characterize *mixed* integer sets. Let $L^x \subseteq \mathbb{R}^p$ be a split polyhedron. We can then use the set $L := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : x \in L^x\}$ for mixed integer sets. We call L a *mixed integer split polyhedron.*

We now consider how to measure the size of a mixed integer split polyhedron. Let $L \subseteq \mathbb{R}^n$ be a mixed integer split polyhedron in $\mathbb{R}^n$ written in the form

$$L := \{x \in \mathbb{R}^n : (\pi^k)^T x \geq \pi_0^k \text{ for } k \in N_f(L)\},$$

where $N_f(L) := \{1, 2, \dots, n_f(L)\}$, $n_f(L)$ denotes the number of facets of L, $(\pi^k, \pi_0^k) \in \mathbb{Z}^{n+1}$ for $k \in N_f(L)$ and $\pi_j^k = 0$ for $j \notin N_I$. We assume that for every $k \in N_f(L)$, π_0^k does not have a common divisor with all the integers π_j^k for $j = 1, 2, \dots, p$. Note that, since L is full dimensional, the representation of L under this assumption is unique.



(a) A polytope P and a split polyhedron L

(b) The only cut that can be derived from L

(c) The strengthened relaxation of P_I

Figure 1: Strengthening a linear relaxation P by using a split polyhedron L

Given a vector $v \in \mathbb{Z}^n$ that satisfies $v_j = 0$ for $j \notin N_I$, the number of parallel hyperplanes $v^T x = v_0$ that intersect a mixed integer split polyhedron $L \subseteq \mathbb{R}^n$ for varying $v_0 \in \mathbb{R}$ gives a measure of how wide L is along the vector v. Define $Z(N_I) := \{v \in \mathbb{Z}^n : v_j = 0 \text{ for all } j \notin N_I\}$. The *width* of L along a vector $v \in Z(N_I)$ is defined to be the number

$$w(L, v) := \max\{v^T x : x \in L\} - \min\{v^T x : x \in L\}.$$

By considering the width of L along all the facets of L, and choosing the largest of these numbers, we obtain a measure of how wide L is.

Definition 2.2 *(The max-facet-width of a mixed integer split polyhedron). Let $L \subseteq \mathbb{R}^n$ be a mixed integer split polyhedron, and let $(\pi^k)^T x \geq \pi_0^k$ denote the facets of L, where $k \in N_f(L)$, $(\pi^k, \pi_0^k) \in \mathbb{Z}^{n+1}$ and $\pi^k \in Z(N_I)$. The* max-facet-width *of L is defined to be the number*

$$w_f(L) := \max\{w(L, \pi^k) : k \in N_f(L)\}.$$

The max-facet-width measures the size of a mixed integer split polyhedron. We now use this measure to also measure the size of a general mixed integer lattice point free rational polyhedron. For this, we use the following result proven in [2] : for every mixed integer lattice point free rational polyhedron $Q \subseteq \mathbb{R}^n$, there exists a mixed integer split polyhedron $L \subseteq \mathbb{R}^n$ that satisfies $\mathrm{ri}(Q) \subseteq \mathrm{int}(L)$. Hence there exists a mixed integer split polyhedron L that excludes at least the same points as Q. A natural measure of the size of Q is then the smallest max-facet-width of a mixed integer split polyhedron with this property.

Definition 2.3 *(Width size of any mixed integer lattice point free rational polyhedron) Let $Q \subseteq \mathbb{R}^n$ be a mixed integer lattice point free rational polyhedron. The* width size *of Q is defined to be the number*

$$\textit{width-size}(Q) := \min\{\textit{max-facet-width}(L) : L \textit{ is a mixed integer split polyhedron s.t. } \mathrm{ri}(Q) \subseteq \mathrm{int}(L)\}.$$

2.1 Polyhedral relaxations from mixed integer split polyhedra As mentioned in the introduction, any mixed integer lattice point free convex set $C \subseteq \mathbb{R}^n$ gives a relaxation of $\mathrm{conv}(P_I)$

$$R(C, P) := \mathrm{conv}(\{x \in P : x \notin \mathrm{ri}(C)\})$$

that satisfies $\mathrm{conv}(P_I) \subseteq R(C, P) \subseteq P$. Since mixed integer split polyhedra L are maximal wrt. inclusion, the sets $R(L, P)$ for mixed integer split polyhedra L are as tight relaxations as possible wrt. this operation. Figure 1 shows the set $R(L, P)$ for a polytope P with five vertices and a split polyhedron L.

For the example in Figure 1, the set $R(L, P)$ is a polyhedron. We now show that, in general, mixed integer split polyhedra give polyhedral relaxations $R(L, P)$ of P_I.

Lemma 2.4 *Let $L \subseteq \mathbb{R}^n$ be a full dimensional polyhedron whose recession cone $0^+(L)$ is a linear space. Then the following set $R(L,P)$ is a polyhedron.*

$$R(L,P) := \text{conv}(\{x \in P : x \notin \text{int}(L)\}).$$

Proof. Let $(l^i)^T x \geq l_0^i$ for $i \in I$ denote the facets of L, where $I := \{1, 2, \ldots, n_f\}$ and n_f denotes the number of facets of L. Also suppose $P = \{x \in \mathbb{R}^n : Dx \leq d\}$, where $D \in \mathbb{Q}^{m \times n}$ and $d \in \mathbb{Q}^m$. Observe that L has the property that, if $x^r \in 0^+(L)$, then $(l^i)^T x^r = 0$ for all $i \in I$. This follows from the fact that the recession cone $0^+(L)$ of L is a linear space. We claim $R(L,P)$ is the projection of the following polyhedron onto the space of x-variables.

$$x = \sum_{i \in I} x^i, \tag{2}$$

$$Dx^i \leq \lambda^i d, \quad \text{for } i \in I, \tag{3}$$

$$(l^i)^T x^i \leq \lambda^i l_0^i, \quad \text{for } i \in I, \tag{4}$$

$$\sum_{i \in I} \lambda^i = 1, \tag{5}$$

$$\lambda^i \geq 0, \quad \text{for } i \in I. \tag{6}$$

The above construction was also used by Balas for disjunctive programming [4]. Let $S(L,P)$ denote the set of $x \in \mathbb{R}^n$ that can be represented in the form (2)-(6) above. We need to prove $R(L,P) = S(L,P)$. A result in Cornuéjols [6] shows that $\text{cl}(\text{conv}(\cup_{i \in I} P^i)) = \text{cl}(R(L,P)) = S(L,P)$, where $P^i := \{x \in \mathbb{R}^n : Dx \leq d \text{ and } (l^i)^T x \leq l_0^i\}$ for $i \in I$. It follows that $R(L,P) \subseteq S(L,P)$, so we only have to show the other inclusion.

We now show $S(L,P) \subseteq R(L,P)$. Let $\bar{x} \in S(L,P)$. By definition this means there exists $\{\bar{x}^i\}_{i \in I}$ and $\{\bar{\lambda}^i\}_{i \in I}$ such that $\bar{x}$, $\{\bar{x}^i\}_{i \in I}$ and $\{\bar{\lambda}^i\}_{i \in I}$ satisfy (2)-(6). Let $\bar{I} := \{i \in I : \bar{x}^i \neq 0\}$. We can assume $|\bar{I}|$ is as small as possible. Furthermore we can assume $|\bar{I}| \geq 2$.

Let $\bar{I}^0 := \{i \in \bar{I} : \bar{\lambda}^i = 0\}$, and let $i_0 \in \bar{I}^0$ be arbitrary. We claim $\bar{x}^{i_0} \in 0^+(R(L,P))$. To show this, we first argue that there exists $i' \in I$ such that $(l^{i'})^T \bar{x}^{i_0} < 0$. Suppose, for a contradiction, that $(l^i)^T \bar{x}^{i_0} \geq 0$ for all $i \in I$. This implies $x^{i_0} \in 0^+(L)$, and therefore $(l^i)^T \bar{x}^{i_0} = 0$ for all $i \in I$. We now show this contradicts the assumption that $|\bar{I}|$ is as small as possible. Indeed, choose $\bar{i} \in \bar{I} \setminus \{i_0\}$ arbitrarily. Define $\tilde{x}^{\bar{i}} := \bar{x}^{i_0} + \bar{x}^{\bar{i}}$, $\tilde{x}^{i_0} := 0$, $\tilde{x}^i := \bar{x}^i$ for $i \in I \setminus \{i_0, \bar{i}\}$ and $\tilde{\lambda}^i := \bar{\lambda}^i$ for $i \in I$. We have that $\bar{x}$, $\{\tilde{x}^i\}_{i \in I}$ and $\{\tilde{\lambda}^i\}_{i \in I}$ satisfy (2)-(6), and $\{\tilde{x}^i\}_{i \in I}$ gives a representation of $\bar{x}$ with fewer non-zero vectors than $\{\bar{x}^i\}_{i \in I}$. This contradicts the minimality of $|\bar{I}|$. Therefore there exists $i' \in I$ such that $(l^{i'})^T \bar{x}^{i_0} < 0$.

We can now show $\bar{x}^{i_0} \in 0^+(R(L,P))$. Let $x^R \in R(L,P)$ be arbitrary and define $\bar{x}^{i_0}(\alpha) := x^R + \alpha \bar{x}^{i_0}$ for $\alpha \geq 0$. Since $(l^{i'})^T \bar{x}^{i_0} < 0$, there exists $\bar{\alpha} > 0$ such that $(l^{i'})^T \bar{x}^{i'_0}(\alpha) \leq l_0^{i'}$ for all $\alpha \geq \bar{\alpha}$. This implies $\bar{x}^{i_0}(\alpha) \in R(L,P)$ for all $\alpha \geq \bar{\alpha}$. Hence $\bar{x}^{i_0} \in 0^+(R(L,P))$.

We can now write $\bar{x} = \sum_{i \in \bar{I}^{>0}} \bar{\lambda}^i \frac{\bar{x}^i}{\bar{\lambda}^i} + \sum_{i \in \bar{I}^0} \bar{x}^i$, where $\bar{I}^{>0} := \{i \in \bar{I} : \bar{\lambda}^i > 0\}$, $\frac{\bar{x}^i}{\bar{\lambda}^i} \in R(L,P)$ for $i \in \bar{I}^{>0}$, $\bar{x}^i \in 0^+(R(L,P))$ for $i \in \bar{I}^0$, $\sum_{i \in \bar{I}^{>0}} \bar{\lambda}^i = 1$ and $\bar{\lambda}^i > 0$ for $i \in \bar{I}^{>0}$. Therefore $\bar{x} \in R(L,P)$. $\square$

Lemma 2.4 implies that, for every *finite* collection $\mathcal{L}$ of mixed integer split polyhedra, the set

$$\text{Cl}(P, \mathcal{L}) := \cap_{L \in \mathcal{L}} R(L,P),$$

is a polyhedron. A next natural question is under which conditions the same is true for an *infinite* collection of mixed integer split polyhedra. As mentioned, we will show that a sufficient condition for this to be the case is that it is possible to provide an upper bound w^* on the max-facet-width of the mixed integer split polyhedra in an infinite collection $\mathcal{L}$ of mixed integer split polyhedra. Therefore, we consider the set of *all* mixed integer split polyhedra whose max-facet-width is bounded by a given constant $w > 0$

$$\mathcal{L}^w := \{L \subseteq \mathbb{R}^n : L \text{ is a mixed integer split polyhedron satisfying } w_f(L) \leq w\}.$$

An extension of the (first) split closure can now be defined.

Definition 2.4 *(The w^{th} split closure).*
Given $w > 0$, the w^{th} split closure *of P is defined to be the set*

$$Cl_w(P, \mathcal{L}^w) := \cap_{L \in \mathcal{L}^w} R(L,P).$$

A natural question is which condition a mixed integer split polyhedron L must satisfy in order to have $R(L,P) \neq P$. The following lemma shows that $R(L,P) \neq P$ exactly when there is a vertex of P in the interior of L.

Lemma 2.5 *Let $L \subset \mathbb{R}^n$ be a mixed integer split polyhedron. Then $R(L,P) \neq P$ if and only if there is a vertex of P in the interior of L.*

Proof. If v^i is a vertex of P in the interior of L, where $i \in V$, then v^i can not be expressed as a convex combination of points in P that are not in the interior of L, and therefore $v^i \notin R(L,P)$. Conversely, when L does *not* contain a vertex of P in its interior, then $\delta^T v^i \geq \delta_0$ for every valid inequality $\delta^T x \geq \delta_0$ for $R(L,P)$ and $i \in V$. Since the extreme rays of $R(L,P)$ are the same as the extreme rays of P, we have $\delta^T r^j \geq 0$ for every extreme ray $j \in E$. □

3. Cutting planes from inner representations of polyhedra The focus in this section is on analyzing the effect of adding cutting planes (or cuts) to the linear relaxation P of P_I. We define cuts to be inequalities that cut off some vertices of P. In other words, we say an inequality $\delta^T x \geq \delta_0$ is a cut for P if $\delta^T v^i < \delta_0$ for some $i \in V$. Let $V^c_{(\delta,\delta_0)} := \{i \in V : \delta^T v^i < \delta_0\}$ index the vertices of P that are cut off by $\delta^T x \geq \delta_0$, and let $V^s_{(\delta,\delta_0)} := \{i \in V : \delta^T v^i \geq \delta_0\}$ index the vertices of P that satisfy $\delta^T x \geq \delta_0$.

A cut $\delta^T x \geq \delta_0$ is called *non-negative* if $\delta^T r^j \geq 0$ for all $j \in E$. Throughout this section we only consider non-negative cutting planes. Observe that non-negativity is a necessary condition for valid cuts for a mixed integer set. Indeed, if $\delta^T x \geq \delta_0$ is a valid cut for the mixed integer points in P, and $j \in E$ is an arbitrary extreme ray of P, then the halfline $\{x^I + \mu r^j : \mu \geq 0\}$ contains an infinite number of mixed integer points for any mixed integer point $x^I \in P$. Therefore, if we had $\delta^T r^j < 0$ for some extreme ray r^j of P, a contradiction to the validity of $\delta^T x \geq \delta_0$ for the mixed integer points in P would be obtained.

3.1 The vertices created by the addition of a cut Adding a non-negative cut $\delta^T x \geq \delta_0$ to the linear relaxation P of P_I creates a polyhedron with different vertices than P. We now analyze the new vertices that are created. For simplicity let $\Lambda := \{\lambda \in \mathbb{R}^{|V|}_+ : \sum_{i \in V} \lambda_i = 1\}$, $\Lambda^c_{(\delta,\delta_0)} := \{\lambda \in \Lambda : \sum_{i \in V^c_{(\delta,\delta_0)}} \lambda_i = 1\}$ and $\Lambda^s_{(\delta,\delta_0)} := \{\lambda \in \Lambda : \sum_{k \in V^s_{(\delta,\delta_0)}} \lambda_k = 1\}$. Also, for any $\lambda \in \Lambda$, define $v_\lambda := \sum_{i \in V} \lambda_i v^i$, and for any $\mu \geq 0$, define $r_\mu := \sum_{j \in E} \mu_j r^j$. We now argue that the new vertices that are created by adding the cut $\delta^T x \geq \delta_0$ to P are *intersection points* [3]. Intersection points are defined as follows. Given an extreme ray $j \in E$ that satisfies $\delta^T r^j > 0$, and a convex combination $\lambda^c \in \Lambda^c_{(\delta,\delta_0)}$, the halfline $\{v_\lambda + \alpha r^j : \alpha \geq 0\}$ intersects the hyperplane $\{x \in \mathbb{R}^n : \delta^T x = \delta_0\}$. For $j \in E$ and $\lambda^c \in \Lambda^c_{(\delta,\delta_0)}$, define

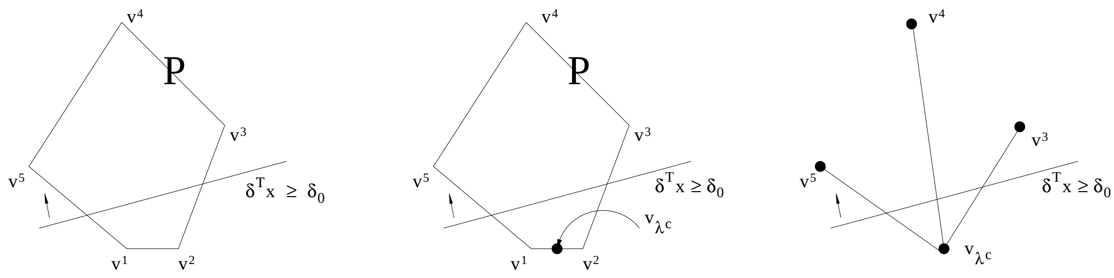
$$\alpha'_j(\delta,\delta_0,\lambda^c) := \begin{cases} \frac{\delta_0 - \delta^T v_{\lambda^c}}{\delta^T r^j} & \text{if } \delta^T r^j > 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

The number $\alpha'_j(\delta,\delta_0,\lambda^c)$ is the value of α for which the point $v_{\lambda^c} + \alpha r^j$ is on the hyperplane $\delta^T x = \delta_0$. When there is no such point, we define $\alpha'_j(\delta,\delta_0,\lambda^c) = +\infty$. If $\alpha'_j(\delta,\delta_0,\lambda^c) < +\infty$, the point $v_{\lambda^c} + \alpha'_j(\delta,\delta_0,\lambda^c) r^j$ is called the intersection point associated with the convex combination $\lambda^c \in \Lambda^c_{(\delta,\delta_0)}$ and the extreme ray r^j of P. Observe that $\alpha'_j(\delta,\delta_0,\lambda^c)$ is linear in λ^c.

Given a convex combination $\lambda^c \in \Lambda^c_{(\delta,\delta_0)}$, and a vertex $k \in V^s_{(\delta,\delta_0)}$, the line segment between v_λ and v^k intersects the hyperplane $\{x \in \mathbb{R}^n : \delta^T x = \delta_0\}$. For $k \in V$ and $\lambda^c \in \Lambda^c_{(\delta,\delta_0)}$, define

$$\beta'_k(\delta,\delta_0,\lambda^c) := \begin{cases} \frac{\delta_0 - \delta^T v_{\lambda^c}}{\delta^T (v^k - v_{\lambda^c})} & \text{if } k \in V^s_{(\delta,\delta_0)}, \\ +\infty & \text{otherwise.} \end{cases} \tag{8}$$

The number $\beta'_k(\delta,\delta_0,\lambda^c)$ denotes the value of β for which the point $v_{\lambda^c} + \beta(v^k - v_{\lambda^c})$ is on the hyperplane $\delta^T x = \delta_0$. Observe that $\beta'_k(\delta,\delta_0,\lambda^c) \in]0,1]$ whenever $\beta'_k(\delta,\delta_0,\lambda^c) < +\infty$. If $\beta'_k(\delta,\delta_0,\lambda^c) < +\infty$, the point $v_{\lambda^c} + \beta'_k(\delta,\delta_0,\lambda^c)(v^k - v_{\lambda^c})$ is called the intersection point associated with the convex combination $\lambda^c \in \Lambda^c_{(\delta,\delta_0)}$ and the vertex v^k of P. For the polytope P of Figure 1 and a cut $\delta^T x \geq \delta_0$, Figure 2 gives an example of how to compute the intersection points for a given convex combination $\lambda^c = (\frac{1}{2}, \frac{1}{2})$.



(a) The polytope P from Figure 1 and a cut with $V^c(\delta, \delta_0) = \{1, 2\}$

(b) The convex combination v_{λ^c} of v^1 and v^2 for $\lambda^c = (\frac{1}{2}, \frac{1}{2})$

(c) The line segments that determine $\beta'_k(\delta, \delta_0, \lambda^c)$ for $k = 3, 4, 5$

Figure 2: Determining the intersection points from a polytope P and a cut $\delta^T x \geq \delta_0$

An important property of an intersection point of the form $v_{\lambda^c} + \beta'_k(\delta, \delta_0, \lambda^c)v^k$ for $k \in V^s_{(\delta,\delta_0)}$ and $\lambda^c \in \Lambda^c_{(\delta,\delta_0)}$ is the following.

Lemma 3.1 *Let $\delta^T x \geq \delta_0$ be a non-negative cut, and let $k \in V^s_{(\delta,\delta_0)}$. For every $\lambda^c \in \Lambda^c_{(\delta,\delta_0)}$, the intersection point $v_{\lambda^c} + \beta'_k(\delta, \delta_0, \lambda^c)(v^k - v_{\lambda^c})$ is a convex combination of the intersection points $v^i + \beta'_k(\delta, \delta_0, e^i)(v^k - v^i)$ for $i \in V^c_{(\delta,\delta_0)}$.*

Proof. Define $C := \text{conv}(\{v^k\} \cup \{v^i\}_{i \in V^c_{(\delta,\delta_0)}})$. Trivially we have $v_{\lambda^c} + \beta'_k(\delta, \delta_0, \lambda^c)(v^k - v_{\lambda^c}) \in C$. We will show the vertices of the polytope $\{x \in C : \delta^T x = \delta_0\}$ are given by the points $v^i + \beta'_k(\delta, \delta_0, e^i)(v^k - v^i)$ for $i \in V^c_{(\delta,\delta_0)}$ from which the result follows. If $\delta^T v^k = \delta_0$, the result is trivial, so we assume $\delta^T v^k > \delta_0$.

Therefore suppose $\bar{x} \in \{x \in C : \delta^T x = \delta_0\}$ is a vertex of $\{x \in C : \delta^T x = \delta_0\}$. We may write $\bar{x} = \lambda_0 v^k + \sum_{i \in V^c_{(\delta,\delta_0)}} \lambda_i v^i$, where $\lambda_0 + \sum_{i \in V^c_{(\delta,\delta_0)}} \lambda_i = 1$, $\lambda_0 \geq 0$ and $\lambda_i \geq 0$ for all $i \in V^c_{(\delta,\delta_0)}$. Using $\lambda_0 = 1 - \sum_{i \in V^c_{(\delta,\delta_0)}} \lambda_i$, we can write $\bar{x} = v^k + \sum_{i \in V^c_{(\delta,\delta_0)}} \lambda_i(v^i - v^k)$. Multiplying with δ on both sides then gives $\sum_{i \in V^c_{(\delta,\delta_0)}} \frac{\lambda_i}{\eta_{i,k}} = 1$, where $\eta_{i,k} := \frac{\delta^T v^k - \delta_0}{\delta^T(v^k - v^i)}$.

We can now write $\bar{x} = v^k + \sum_{i \in V^c_{(\delta,\delta_0)}} \lambda_i(v^i - v^k) = \sum_{i \in V^c_{(\delta,\delta_0)}} \frac{\lambda_i}{\eta_{i,k}} v^k + \sum_{i \in V^c_{(\delta,\delta_0)}} \lambda_i(v^i - v^k) = \sum_{i \in V^c_{(\delta,\delta_0)}} \frac{\lambda_i}{\eta_{i,k}}(v^k + \eta_{i,k}(v^i - v^k))$. Since $v^k + \eta_{i,k}(v^i - v^k) = v^i + \beta'_k(\delta, \delta_0, e^i)(v^k - v^i)$ for $i \in V^c_{(\delta,\delta_0)}$, the result follows. □

Lemma 3.1 shows that the only vectors $\lambda^c \in \Lambda^c_{(\delta,\delta_0)}$ for which the intersection points of the type $v_{\lambda^c} + \beta'_k(\delta, \delta_0, \lambda^c)(v^k - v_{\lambda^c})$ can be vertices of $\{x \in P : \delta^T x \geq \delta_0\}$ are the unit vectors.

In order to characterize the vertices of $\{x \in P : \delta^T x \geq \delta_0\}$, we first give a representation of $\{x \in P : \delta^T x \geq \delta_0\}$ in a higher dimensional space. Note that any point which is a convex combination of the vertices of P can be written as a convex combination of two points v_{λ^s} and v_{λ^c}, where $\lambda^s \in \Lambda^s_{(\delta,\delta_0)}$ and $\lambda^c \in \Lambda^c_{(\delta,\delta_0)}$. We may therefore write P in the form

$$P = \{x \in \mathbb{R}^n : x = x^v + r_\mu, \mu \geq 0, \lambda^s \in \Lambda^s_{(\delta,\delta_0)}, \lambda^c \in \Lambda^c_{(\delta,\delta_0)} \text{ and } x^v \in \text{conv}(v_{\lambda^c}, v_{\lambda^s})\}.$$

Consider the set obtained from P by fixing the convex combination $\lambda^c \in \Lambda^c_{(\delta,\delta_0)}$

$$P(\lambda^c) := \{x \in \mathbb{R}^n : x = x^v + r_\mu, \mu \geq 0, \lambda^s \in \Lambda^s_{(\delta,\delta_0)} \text{ and } x^v \in \text{conv}(v_{\lambda^c}, v_{\lambda^s})\}.$$

Observe that we may write $P(\lambda^c)$ in the form

$$P(\lambda^c) = \{x \in \mathbb{R}^n : x = v_{\lambda^c} + \sum_{k \in V^s_{(\delta,\delta_0)}} \epsilon_k(v^k - v_{\lambda^c}) + r_\mu, \mu \geq 0 \text{ and } \epsilon \in \Lambda\}.$$

Now consider the set $P^l(\lambda^c)$ obtained from $P(\lambda^c)$ by also considering the multipliers on the vertices of P indexed by $V^s_{(\delta,\delta_0)}$, and the extreme rays of P

$$P^l(\lambda^c) := \{(x, \epsilon, \mu) \in \mathbb{R}^{n+|V|+|E|} : x = v_\lambda + \sum_{k \in V^s_{(\delta,\delta_0)}} \epsilon_k(v^k - v_{\lambda^c}) + r_\mu, \mu \geq 0 \text{ and } \epsilon \in \Lambda\}.$$

The scalars $\alpha'_j(\delta,\delta_0,\lambda^c)$ for $j\in E$ and $\beta'_i(\delta,\delta_0,\lambda^c)$ for $i\in V$ give an alternative description of the set of points in P that satisfy $\delta^T x\geq\delta_0$ in a higher dimensional space.

Lemma 3.2 *(Lemma 2 in [1]). Let $\delta^T x\geq\delta_0$ be a non-negative cut for P. For any $\lambda^c\in\Lambda^c_{(\delta,\delta_0)}$, we have*

$$\{(x,\epsilon,\mu)\in P^l(\lambda^c):\delta^T x\geq\delta_0\}=\{(x,\epsilon,\mu)\in P^l(\lambda^c):\sum_{j\in E}\frac{\mu_j}{\alpha'_j(\delta,\delta_0,\lambda^c)}+\sum_{k\in V}\frac{\epsilon_k}{\beta'_k(\delta,\delta_0,\lambda^c)}\geq 1\}.$$

Proof. We have $(\bar{x},\bar{\epsilon},\bar{\mu})\in P^l(\lambda^c)$ and $\delta^T\bar{x}\geq\delta_0 \iff \bar{x}=v_{\lambda^c}+\sum_{k\in V^s_{(\delta,\delta_0)}}\bar{\epsilon}_k(v^k-v_{\lambda^c})+r_{\bar{\mu}}$, where $\bar{\epsilon},\bar{\mu}\geq 0$, $\sum_{k\in V^s_{(\delta,\delta_0)}}\bar{\epsilon}_k\leq 1$ and $\delta^T\bar{x}\geq\delta_0 \iff \bar{x}=v_{\lambda^c}+\sum_{k\in V^s_{(\delta,\delta_0)}}\bar{\epsilon}_k(v^k-v_{\lambda^c})+r_{\bar{\mu}}$, $\bar{\epsilon},\bar{\mu}\geq 0$, $\sum_{k\in V^s_{(\delta,\delta_0)}}\bar{\epsilon}_k\leq 1$ and $\sum_{k\in V^s_{(\delta,\delta_0)}}\bar{\epsilon}_k\delta^T(v^k-v_{\lambda^c})+\sum_{j\in E}\bar{\mu}_j(\delta^T r^j)\geq(\delta_0-\delta^T v_{\lambda^c}) \iff (\bar{x},\bar{\epsilon},\bar{\mu})\in P^l(\lambda^c)$ and $\sum_{k\in V}\bar{\epsilon}_k/\beta'_k(\delta,\delta_0,\lambda^c)+\sum_{j\in E}\bar{\mu}_j/\alpha'_j(\delta,\delta_0,\lambda^c)\geq 1$. □

Based on the above result, we can now characterize the vertices of $\{x\in P:\delta^T x\geq\delta_0\}$. Specifically we show that every vertex of $\{x\in P:\delta^T x\geq\delta_0\}$ is either a vertex of P that satisfies $\delta^T x\geq\delta_0$, or an intersection point obtained from a vertex of P that violates $\delta^T x\geq\delta_0$.

Lemma 3.3 *Let $\delta^T x\geq\delta_0$ be a non-negative cut for P. The vertices of $\{x\in P:\delta^T x\geq\delta_0\}$ are:*

(i) vertices v^k of P with $k\in V^s_{(\delta,\delta_0)}$,

(ii) intersection points $v^i+\beta'_k(\delta,\delta_0,e^i)(v^k-v^i)$, where $i\in V^c_{(\delta,\delta_0)}$ and $k\in V^s_{(\delta,\delta_0)}$, and

(iii) intersection points $v^i+\alpha'_j(\delta,\delta_0,e^i)r^j$, where $i\in V^c_{(\delta,\delta_0)}$ and $j\in E$ satisfies $\delta^T r^j>0$.

Proof. Let $\bar{x}\in\{x\in P:\delta^T x\geq\delta_0\}$ be a vertex of $\{x\in P:\delta^T x\geq\delta_0\}$. Also let $\lambda^c\in\Lambda^c_{(\delta,\delta_0)}$ and $(\bar{\epsilon},\bar{\mu})$ be such that $(\bar{x},\bar{\epsilon},\bar{\mu})\in P^l(\lambda^c)$ and $\bar{x}=v_{\lambda^c}+\sum_{k\in V^s_{(\delta,\delta_0)}}\bar{\epsilon}_k(v^k-v_{\lambda^c})+\sum_{j\in E}\bar{\mu}_j r^j$. Since $P(\lambda^c)\subseteq P$, we must have that $\bar{x}$ is a vertex of $\{x\in P(\lambda^c):\delta^T x\geq\delta_0\}$. We first show that $\bar{x}$ must be either of the form: (a) a vertex v^k of P with $k\in V^s_{(\delta,\delta_0)}$, (b) an intersection point $v_{\lambda^c}+\beta'_k(\delta,\delta_0,\lambda^c)(v^k-v_{\lambda^c})$ with $k\in V^s_{(\delta,\delta_0)}$, or (c) an intersection point $v_{\lambda^c}+\alpha'_j(\delta,\delta_0,\lambda^c)r^j$ with $j\in E$ satisfying $\delta^T r^j>0$.

Clearly, if $\bar{x}$ is a vertex of $\{x\in P(\lambda^c):\delta^T x\geq\delta_0\}$ which is *not* a vertex of P, we must have that $\bar{x}$ satisfies $\delta^T x\geq\delta_0$ with equality. From $\delta^T\bar{x}=\delta_0$, it follows from Lemma 3.2 that $(\bar{x},\bar{\epsilon},\bar{\mu})\in P^l(\lambda^c)$ and

$$\sum_{j\in E}\frac{\bar{\mu}_j}{\alpha'_j(\delta,\delta_0,\lambda^c)}+\sum_{k\in V^s_{(\delta,\delta_0)}}\frac{\bar{\epsilon}_k}{\beta'_k(\delta,\delta_0,\lambda^c)}=1.$$

We can now write

$$\bar{x}=\sum_{j\in E\setminus E^0}\eta_j(v_{\lambda^c}+\alpha'_j(\delta,\delta_0,\lambda^c)r^j)+\sum_{k\in V^s_{(\delta,\delta_0)}}\gamma_k(v_{\lambda^c}+\beta'_k(\delta,\delta_0,\lambda^c)(v^k-v_{\lambda^c}))+\sum_{j\in E^0}\bar{\mu}_j r^j,$$

where $E^0:=\{j\in E:\delta^T r^j=0\}$, $\eta_j:=\frac{\bar{\mu}_j}{\alpha'_j(\delta,\delta_0,\lambda^c)}$ for $j\in E\setminus E^0$, $\gamma_k:=\frac{\bar{\epsilon}_k}{\beta'_k(\delta,\delta_0,\lambda^c)}$ for $k\in V^s_{(\delta,\delta_0)}$ and $\sum_{j\in E\setminus E^0}\eta_j+\sum_{k\in V^s_{(\delta,\delta_0)}}\gamma_k=1$. Hence $\bar{x}$ must be of one of the forms (a)-(c) above.

We now show (i)-(iii). If $\bar{x}$ is a vertex v^k of P, where $k\in V^s_{(\delta,\delta_0)}$, we are done, so we may assume that either $\bar{x}=v_{\lambda^c}+\beta'_k(\delta,\delta_0,\lambda^c)(v^k-v_{\lambda^c})$, where $k\in V^s_{(\delta,\delta_0)}$, or $\bar{x}=v_{\lambda^c}+\alpha'_j(\delta,\delta_0,\lambda^c)r^j$, where $j\in E$ satisfies $\alpha'_j(\delta,\delta_0,\lambda^c)<+\infty$. If $\bar{x}$ is of the form $\bar{x}=v_{\lambda^c}+\alpha'_j(\delta,\delta_0,\lambda^c)r^j$, we may write $\bar{x}=v_{\lambda^c}+\alpha'_j(\delta,\delta_0,\lambda^c)r^j=v_{\lambda^c}+\frac{\delta_0-\delta^T v_{\lambda^c}}{\delta^T r^j}r^j=\sum_{i\in V^c_{(\delta,\delta_0)}}\lambda_i(v^i+\frac{\delta_0-\delta^T v^i}{\delta^T r^j}r^j)$. Since $\alpha'_j(\delta,\delta_0,e^i)=\frac{\delta_0-\delta^T v^i}{\delta^T r^j}$ and $\bar{x}$ is a vertex of $\{x\in P:\delta^T x\geq\delta_0\}$, this implies $\lambda_{\bar{i}}=1$ for some $\bar{i}\in V^c_{(\delta,\delta_0)}$. Finally, if $\bar{x}$ is of the form $\bar{x}=v_{\lambda^c}+\beta'_k(\delta,\delta_0,\lambda^c)(v^k-v_{\lambda^c})$, then Lemma 3.1 shows that $\bar{x}$ is of the form $v^{\bar{i}}+\beta'_k(\delta,\delta_0,e^{\bar{i}})(v^k-v^{\bar{i}})$ for some $\bar{i}\in V^c_{(\delta,\delta_0)}$ and $k\in V^s_{(\delta,\delta_0)}$. □

Lemma 3.3 motivates the following notation for those intersection points $v_{\lambda^c}+\alpha'_j(\delta,\delta_0,\lambda^c)r^j$ and $v_{\lambda^c}+\beta'_k(\delta,\delta_0,\lambda^c)(v^k-v_{\lambda^c})$, where λ^c is a unit vector. Given $i\in V^c_{(\delta,\delta_0)}$ and $j\in E$, define $\alpha'_{i,j}(\delta,\delta_0):=\alpha'_j(\delta,\delta_0,e^i)$, and given $i\in V^c_{(\delta,\delta_0)}$ and $k\in V^s_{(\delta,\delta_0)}$, define $\beta'_{i,k}(\delta,\delta_0):=\beta'_k(\delta,\delta_0,e^i)$.

3.2 Dominance and equivalence between cuts Given two non-negative cuts $(\delta^1)^T x \geq \delta_0^1$ and $(\delta^2)^T x \geq \delta_0^2$ for P, it is not clear how to compare them in the space of the x variables. By including the multipliers on the extreme rays and on the satisfied vertices in the description, such a comparison is possible. We assume all non-negative cuts considered in this section all cut off exactly the same set of vertices $V^c \subseteq V$ of P. Our notion of dominance is the following.

Definition 3.1 *Let $(\delta^1)^T x \geq \delta_0^1$ and $(\delta^2)^T x \geq \delta_0^2$ be two non-negative cuts for P that cut off the same set om vertices $V^c = V^c_{(\delta^1,\delta^1_0)} = V^c_{(\delta^2,\delta^2_0)}$ of P.*

(i) The cutting plane $(\delta^1)^T x \geq \delta_0^1$ dominates *$(\delta^2)^T x \geq \delta_0^2$ on P iff $\{x \in P : (\delta^1)^T x \geq \delta_0^1\} \subseteq \{x \in P : (\delta^2)^T x \geq \delta_0^2\}$.*

(ii) If $(\delta^1)^T x \geq \delta_0^1$ dominates $(\delta^2)^T x \geq \delta_0^2$ on P, and $(\delta^2)^T x \geq \delta_0^2$ dominates $(\delta^1)^T x \geq \delta_0^1$ on P, we say $(\delta^1)^T x \geq \delta_0^1$ and $(\delta^2)^T x \geq \delta_0^2$ are equivalent *on P.*

We now show that an equivalent definition of dominance between a pair of non-negative cuts is possible, which is based on intersection points.

Lemma 3.4 *Let $(\delta^1)^T x \geq \delta_0^1$ and $(\delta^2)^T x \geq \delta_0^2$ be non-negative cuts for P satisfying $V^c = V^c_{(\delta^1,\delta^1_0)} = V^c_{(\delta^2,\delta^2_0)}$. Then $(\delta^1)^T x \geq \delta_0^1$ dominates $(\delta^2)^T x \geq \delta_0^2$ on P if and only if*

(i) The inequality $\frac{1}{\alpha'_{i,j}(\delta^1,\delta^1_0)} \leq \frac{1}{\alpha'_{i,j}(\delta^2,\delta^2_0)}$ holds for $j \in E$ and $i \in V^c$.
(The halfline $\{v^i + \alpha r^j : \alpha \geq 0\}$ is intersected later by $(\delta^1)^T x \geq \delta_0^1$ than $(\delta^2)^T x \geq \delta_0^2$)

(ii) The inequality $\frac{1}{\beta'_{i,k}(\delta^1,\delta^1_0)} \leq \frac{1}{\beta'_{i,k}(\delta^2,\delta^2_0)}$ holds for $k \in V \setminus V^c$ and $i \in V^c$.
(The halfline $\{v^i + \beta(v^k - v^i) : \beta \geq 0\}$ is intersected later by $(\delta^1)^T x \geq \delta_0^1$ than $(\delta^2)^T x \geq \delta_0^2$)

Proof. Define $Q^1 := \{x \in P : (\delta^1)^T x \geq \delta_0^1\}$ and $Q^2 := \{x \in P : (\delta^2)^T x \geq \delta_0^2\}$. First suppose $(\delta^1)^T x \geq \delta_0^1$ dominates $(\delta^2)^T x \geq \delta_0^2$ on P, *i.e.*, suppose $Q^1 \subseteq Q^2$. We will verify that (i) and (ii) are satisfied. First let $i \in V^c$ and $j \in E$ be arbitrary. If $\alpha'_{i,j}(\delta^1,\delta^1_0) = +\infty$, then trivially $0 = \frac{1}{\alpha'_{i,j}(\delta^1,\delta^1_0)} \leq \frac{1}{\alpha'_{i,j}(\delta^2,\delta^2_0)}$. If $\alpha'_{i,j}(\delta^1,\delta^1_0) < +\infty$, then the intersection point $\bar{y} := v^i + \alpha'_{i,j}(\delta^1,\delta^1_0) r^j$ satisfies $(\delta^1)^T \bar{y} = \delta^1_0$, and therefore $\bar{y} \in Q^1 \subseteq Q^2$. Hence we have $(\delta^2)^T \bar{y} = (\delta^2)^T v^i + \alpha'_{i,j}(\delta^1,\delta^1_0)(\delta^2)^T r^j \geq \delta^2_0$, which implies $\frac{1}{\alpha'_{i,j}(\delta^1,\delta^1_0)} \leq \frac{1}{\alpha'_{i,j}(\delta^2,\delta^2_0)}$.

Now let $i \in V^c$ and $k \in V \setminus V^c$ be arbitrary. The point $\bar{z} := v^i + \beta'_{i,k}(\delta^1,\delta^1_0)(v^k - v^i)$ satisfies $(\delta^1)^T \bar{z} = \delta^1_0$. Hence $\bar{z} \in Q^1 \subseteq Q^2$, and therefore $(\delta^2)^T \bar{z} = (\delta^2)^T v^i + \beta'_{i,k}(\delta^1,\delta^1_0)(\delta^2)^T(v^k - v^i) \geq \delta^2_0$, which implies $\frac{1}{\beta'_{i,k}(\delta^1,\delta^1_0)} \leq \frac{1}{\beta'_{i,k}(\delta^2,\delta^2_0)}$.

Conversely suppose (i) and (ii) are satisfied. Since $V^c = V^c_{(\delta^1,\delta^1_0)} = V^c_{(\delta^2,\delta^2_0)}$, every vertex v^k of P with $k \in V \setminus V^c$ is a vertex of both Q^1 and Q^2. Furthermore, (i) ensures that every vertex of Q^1 of the form $v^i + \alpha'_{i,j}(\delta^1,\delta^1_0) r^j$ belongs to Q^2, where $i \in V^c$, $j \in E$ and $\alpha'_{i,j}(\delta^1,\delta^1_0) < +\infty$. Finally, (ii) ensures that every vertex of Q^1 of the form $v^i + \beta'_{i,k}(\delta^1,\delta^1_0)(v^k - v^i)$ belongs to Q^2, where $i \in V^c$ and $k \in V \setminus V^c$. We have therefore shown that every vertex of Q^1 belongs to Q^2. Since the sets Q^1 and Q^2 have the same extreme rays $\{r^j\}_{j\in E}$, we therefore have $Q^1 \subseteq Q^2$. □

Let $V^c \subseteq V$ be arbitrary, and let $\{(\delta^1)^T x \geq \delta^1_0\}_{l=1}^m$ a finite set of non-negative cuts. We assume $V^c(\delta^l, \delta^l_0) = V^c$ for all $l \in \{1,2,\ldots,m\}$. We now derive a dominance result for the polyhedron $Q(V^c)$

$$Q(V^c) := \{x \in P : (\delta^l)^T x \geq \delta^l_0 \text{ for } l = 1,2,\ldots,m\}.$$

Lemma 3.5 *(This lemma is a generalization of Lemma 3 in [1])*
Assume $Q(V^c) \neq \emptyset$. Let $\delta^T x \geq \delta_0$ be a non-negative cut for P satisfying $V^c(\delta,\delta_0) = V^c$. Then $\delta^T x \geq \delta_0$ is valid for $Q(V^c)$ iff there exists a non-negative cut $(\delta')^T x \geq \delta'_0$ for P that satisfies

(i) $(\delta')^T x \geq \delta'_0$ is a convex combination of the inequalities $(\delta^l)^T x \geq \delta^l_0$ for $l = 1,2,\ldots,m$,

(ii) $(\delta')^T x \geq \delta'_0$ dominates $\delta^T x \geq \delta_0$ on P.

Proof. Consider the linear program (LP) given by $\min\{\delta^T x : x \in Q(V^c)\}$. The assumption $Q(V^c) \neq \emptyset$ and the validity of $\delta^T x \geq \delta_0$ for $Q(V^c)$ implies that (LP) is feasible and bounded. We can formulate (LP) as follows.

$$\begin{aligned}
&\min\ \delta^T x \\
&x = \sum_{i \in \mathrm{V}^c} \lambda_i v^i + \sum_{i \in \mathrm{V}^c} \sum_{k \in V \setminus \mathrm{V}^c} \epsilon^i_k (v^k - v^i) + \sum_{i \in \mathrm{V}^c} \sum_{j \in E} \mu^i_j r^j, && (u)\\
&(\delta^l)^T x \geq \delta^l_0 \text{ for all } l \in \{1, 2, \ldots, m\}, && (w_l)\\
&\sum_{k \in V \setminus \mathrm{V}^c} \epsilon^i_k \leq \lambda_i \text{ for all } i \in \mathrm{V}^c, && (z_i)\\
&\sum_{i \in \mathrm{V}^c} \lambda_i = 1, && (u_0)\\
&\epsilon^i, \mu^i, \lambda \geq 0 \text{ for all } i \in \mathrm{V}^c.
\end{aligned}$$

From the dual of (LP), we obtain $\bar{u} \in \mathbb{R}^n$, $\bar{w} \in \mathbb{R}^m$, $\bar{z} \in \mathbb{R}^{|\mathrm{V}^c|}$ and $\bar{u}_0 \in \mathbb{R}$ that satisfy

(i) $\bar{u}_0 + \sum_{l=1}^m \bar{w}_l \delta^l_0 \geq \delta_0$,
(ii) $-\bar{u} = \delta$,
(iii) $\bar{u}^T v^i + \sum_{l=1}^m \bar{w}_l (\delta^l)^T v^i + \bar{z}_i + \bar{u}_0 \leq 0$ for all $i \in \mathrm{V}^c$.
(iv) $\bar{u}^T (v^k - v^i) + \sum_{l=1}^m \bar{w}_l (\delta^l)^T (v^k - v^i) - \bar{z}_i \leq 0$ for all $i \in \mathrm{V}^c$ and $k \in V \setminus \mathrm{V}^c$.
(v) $\bar{u}^T r^j + \sum_{l=1}^m \bar{w}_l (\delta^l)^T r^j \leq 0$ for all $j \in E$.
(vi) $\bar{w} \geq 0$ and $\bar{z} \geq 0$.

Let $\bar{\delta} := \sum_{l=1}^m \bar{w}_l \delta^l$ and $\bar{\delta}_0 := \sum_{l=1}^m \delta^l_0 \bar{w}_l$. Since $\bar{\delta}^T x \geq \bar{\delta}_0$ is a non-negative combination of the inequalities $\{(\delta^l)^T x \geq \delta^l_0\}_{l=1}^m$, we have that $\bar{\delta}^T x \geq \bar{\delta}_0$ is valid for $Q(V^c)$. Furthermore, the inequality $\bar{\delta}^T x \geq \bar{\delta}_0$ is a non-negative combination of non-negative cuts for P, and therefore $\bar{\delta}^T x \geq \bar{\delta}_0$ is also a non-negative cut for P. Finally, since $V^c(\delta^l, \delta^l_0) = V^c$ for all $l \in \{1, 2, \ldots, m\}$, we have $V^c(\bar{\delta}, \bar{\delta}_0) = V^c$. We will show that $\bar{\delta}^T x \geq \bar{\delta}_0$ dominates $\delta^T x \geq \delta_0$ on P. The system (i)-(vi) implies the following inequalities.

(a) $\bar{u}_0 + \bar{\delta}_0 \geq \delta_0$.
(b) $-\delta^T v^i + \bar{\delta}^T v^i + \bar{z}_i + \bar{u}_0 \leq 0$ for all $i \in \mathrm{V}^c$.
(c) $-\delta^T (v^k - v^i) + \bar{\delta}^T (v^k - v^i) - \bar{z}_i \leq 0$ for all $i \in \mathrm{V}^c$ and $k \in V \setminus \mathrm{V}^c$.
(d) $-\delta^T r^j + \bar{\delta}^T r^j \leq 0$ for all $j \in E$.
(e) $\bar{w} \geq 0$ and $\bar{z} \geq 0$.

We first show $\frac{1}{\alpha'_{i,j}(\bar{\delta},\bar{\delta}_0)} \leq \frac{1}{\alpha'_{i,j}(\delta,\delta_0)}$ for all $i \in V^c$ and $j \in E$. Therefore let $\bar{i} \in V^c$ and $\bar{j} \in E$. If $\alpha'_{\bar{i},\bar{j}}(\delta, \delta_0) = +\infty$, then $\delta^T r^{\bar{j}} = 0$, which by (d) implies that also $\bar{\delta}^T r^{\bar{j}} = 0$, and therefore $0 = \frac{1}{\alpha'_{\bar{i},\bar{j}}(\bar{\delta},\bar{\delta}_0)} = \frac{1}{\alpha'_{\bar{i},\bar{j}}(\delta,\delta_0)}$. Furthermore, if $\bar{\delta}^T r^{\bar{j}} = 0$, then trivially $0 = \frac{1}{\alpha'_{\bar{i},\bar{j}}(\bar{\delta},\bar{\delta}_0)} \leq \frac{1}{\alpha'_{\bar{i},\bar{j}}(\delta,\delta_0)}$. We can therefore assume $\alpha'_{\bar{i},\bar{j}}(\delta, \delta_0) < +\infty$ and $\bar{\delta}^T r^{\bar{j}} > 0$. Multiplying the inequality of (d) corresponding to $\bar{j}$ with $\alpha'_{\bar{i},\bar{j}}(\delta, \delta_0)$ and adding the result to the inequality of (b) corresponding to $\bar{i}$ gives $-\delta^T (v^{\bar{i}} + \alpha'_{\bar{i},\bar{j}}(\delta, \delta_0) r^{\bar{j}}) + \bar{\delta}^T (v^{\bar{i}} + \alpha'_{\bar{i},\bar{j}}(\delta, \delta_0) r^{\bar{j}}) \leq -\bar{u}_0 - \bar{z}_{\bar{i}} \leq \bar{\delta}_0 - \delta_0$. Since we have $\delta^T (v^{\bar{i}} + \alpha'_{\bar{i},\bar{j}}(\delta, \delta_0) r^{\bar{j}}) = \delta_0$, this implies $\bar{\delta}^T (v^{\bar{i}} + \alpha'_{\bar{i},\bar{j}}(\delta, \delta_0) r^{\bar{j}}) \leq \bar{\delta}_0$. Now, $\alpha'_{\bar{i},\bar{j}}(\bar{\delta}, \bar{\delta}_0)$ is defined as the smallest value of α such that $\bar{\delta}^T (v^{\bar{i}} + \alpha r^{\bar{j}}) = \bar{\delta}_0$. Since $\bar{\delta}^T (v^{\bar{i}} + \alpha'_{\bar{i},\bar{j}}(\delta, \delta_0) r^{\bar{j}}) \leq \bar{\delta}_0$, this means we must have $\alpha'_{\bar{i},\bar{j}}(\bar{\delta}, \bar{\delta}_0) \geq \alpha'_{\bar{i},\bar{j}}(\delta, \delta_0)$, and therefore $\frac{1}{\alpha'_{\bar{i},\bar{j}}(\bar{\delta},\bar{\delta}_0)} \leq \frac{1}{\alpha'_{\bar{i},\bar{j}}(\delta,\delta_0)}$. Hence condition (i) of Lemma 3.4 is satisfied.

We now show $\frac{1}{\beta'_{i,k}(\bar{\delta},\bar{\delta}_0)} \leq \frac{1}{\beta'_{i,k}(\delta,\delta_0)}$ for all $i \in V^c$ and $k \in V \setminus V^c$. Therefore let $\bar{i} \in V^c$ and $\bar{k} \in V \setminus V^c$. Multiplying the inequality of (c) corresponding to $(\bar{i}, \bar{k})$ with $\beta'_{\bar{i},\bar{k}}(\delta, \delta_0)$ and adding the result to the inequality of (b) corresponding to $\bar{i}$ gives $-\delta^T (v^{\bar{i}} + \beta'_{\bar{i},\bar{k}}(\delta, \delta_0)(v^{\bar{k}} - v^{\bar{i}})) + \bar{\delta}^T (v^{\bar{i}} + \beta'_{\bar{i},\bar{k}}(\delta, \delta_0)(v^{\bar{k}} - v^{\bar{i}})) \leq -\bar{u}_0 - \bar{z}_{\bar{i}} \leq \bar{\delta}_0 - \delta_0$. Since $\delta^T (v^{\bar{i}} + \beta'_{\bar{i},\bar{k}}(\delta, \delta_0)(v^{\bar{k}} - v^{\bar{i}})) = \delta_0$, this implies $\bar{\delta}^T (v^{\bar{i}} + \beta'_{\bar{i},\bar{k}}(\delta, \delta_0)(v^{\bar{k}} - v^{\bar{i}})) \leq \bar{\delta}_0$. We have that $\beta'_{\bar{i},\bar{k}}(\bar{\delta}, \bar{\delta}_0)$ is defined as the smallest value of β s.t. $\bar{\delta}^T (v^{\bar{i}} + \beta(v^{\bar{k}} - v^{\bar{i}})) = \bar{\delta}_0$, and since

$\bar{\delta}^T(v^{\bar{i}} + \alpha'_{\bar{i},\bar{j}}(\delta,\delta_0)r^{\bar{j}}) \le \bar{\delta}_0$, this implies $\beta'_{\bar{i},\bar{k}}(\bar{\delta},\bar{\delta}_0) \ge \beta'_{\bar{i},\bar{k}}(\delta,\delta_0)$. It follows that $\frac{1}{\beta'_{\bar{i},\bar{k}}(\bar{\delta},\bar{\delta}_0)} \le \frac{1}{\beta'_{\bar{i},\bar{k}}(\delta,\delta_0)}$. Hence condition (ii) of Lemma 3.4 is also satisfied, and therefore $\bar{\delta}^T x \ge \bar{\delta}_0$ dominates $\delta^T x \ge \delta_0$ on P.

To finish the proof, we will argue that we can choose $\bar{\delta}^T x \ge \bar{\delta}_0$ to be a convex combination of the inequalities $\{(\delta^l)^T x \ge \delta^l_0\}_{l=1}^m$. Observe that, if $\sum_{l=1}^m \bar{w}_l \ne 0$, then the inequality $(\delta')^T x \ge \delta'_0$ defined by $(\delta',\delta'_0) := \frac{1}{\sum_{l=1}^m \bar{w}_l}(\bar{\delta},\bar{\delta}_0)$ is a convex combination of the inequalities $\{(\delta^l)^T x \ge \delta^l_0\}_{l=1}^m$ and $(\delta')^T x \ge \delta'_0$ is equivalent to $\bar{\delta}^T x \ge \bar{\delta}_0$ on P. We therefore only have to show $\sum_{l=1}^m \bar{w}_l \ne 0$. If $\sum_{l=1}^m \bar{w}_l = 0$, then (i)-(iii) give $\bar{u}_0 \ge \delta_0$ and $-\delta^T v^i + \bar{z}_i + \bar{u}_0 \le 0$ for all $i \in V^c$, which implies $\delta^T v^i \ge \delta_0$ for all $i \in V^c$. Furthermore, (iv) reads $-\delta^T(v^k - v^i) - \bar{z}_i \le 0$ for all $i \in V^c$ and $k \in V \setminus V^c$. Given $\bar{i} \in V^c$ and $\bar{k} \in V \setminus V^c$, adding the inequality $-\delta^T(v^{\bar{k}} - v^{\bar{i}}) - \bar{z}_{\bar{i}} \le 0$ of (iv) to the inequality of (iii) corresponding to $\bar{i}$ gives $-\delta^T v^{\bar{k}} \le -\bar{u}_0 \le -\delta_0$. Hence $\delta^T x \ge \delta_0$ is satisfied by all vertices of P, which contradicts that $\delta^T x \ge \delta_0$ is a cut for P. Hence $\sum_{l=1}^m \bar{w}_l \ne 0$. $\square$

3.3 A sufficient condition for polyhedrality We now consider the addition of an *infinite* family of non-negative cuts to the polyhedron P. Specifically, consider the convex set

$$X := \{x \in P : (\delta^l)^T x \ge \delta^l_0 \text{ for } l \in I\},$$

where I is now allowed to be an *infinite* index set. The goal in this section is to provide a sufficient condition for X to be a polyhedron. For this purpose, we can assume $V^c(\delta^l,\delta^l_0) = V^c$ for all $l \in I$, *i.e.*, we can assume all cuts cut off the same vertices. Indeed, if the cuts $l \in I$ do not cut off the same set of vertices, then define the set

$$I^c(S) := \{l \in I : V^c(\delta^l,\delta^l_0) = S\}$$

for every $S \subseteq V$, and let $\mathcal{S} := \{S \subseteq V : I^c(S) \ne \emptyset\}$. We can then write

$$X = \cap_{S\in\mathcal{S}}\{x \in P : (\delta^l)^T x \ge \delta^l_0 \text{ for } l \in S\}$$

Since $\mathcal{S}$ is finite, we have that X is a polyhedron if and only if X is a polyhedron under the assumption that $V^c(\delta^l,\delta^l_0) = V^c$ for all $l \in I$.

For simplicity let $\alpha'_{i,j,l} := \alpha'_{i,j}(\delta^l,\delta^l_0)$ for all $(i,j,l) \in V^c \times E \times I$, and $\beta'_{i,k,l} := \beta'_{i,k}(\delta^l,\delta^l_0)$ for all $(i,k,l) \in V^c \times (V \setminus V^c) \times I$. Furthermore, for any $l \in I$, let $\alpha'_{.l}$ denote the vector in $\mathbb{R}^{|V^c|\times|E|}$ whose coordinates are $\alpha'_{i,j,l}$ for $(i,j) \in V^c \times E$, and let $\beta'_{.l}$ denote the vector in $\mathbb{R}^{|V^c|\times|V\setminus V^c|}$ whose coordinates are $\beta'_{i,k,l}$ for $(i,k) \in V^c \times (V \setminus V^c)$.

We will show that X is a polyhedron when the following assumption holds.

Assumption 3.1 *Let $\alpha^* > 0$ and $\beta^* \in]0,1]$ be arbitrary.*

(1) For all $(i,j) \in V^c \times E$, the set $IP^e_{(i,j)}(\alpha^*) := \{\alpha'_{i,j,l} \ge \alpha^* : l \in I\}$ is finite
(There is only a finite number of intersection points between the inequalities $(\delta^l)^T x \ge \delta^l_0$ for $l \in I$ and the halfline $\{v^i + \alpha r^j : \alpha \ge \alpha^*\}$).

(2) For all $(i,k) \in V^c \times V \setminus V^c$, the set $IP^v_{(i,k)}(\beta^*) := \{\beta'_{i,k,l} \ge \beta^* : l \in I\}$ is finite
(There is only a finite number of intersection points between the inequalities $(\delta^l)^T x \ge \delta^l_0$ for $l \in I$ and the halfline $\{v^i + \beta(v^k - v^i) : \beta \ge \beta^*\}$).

The main theorem is the following.

Theorem 3.1 *Suppose $\{(\delta^l)^T x \ge \delta^l_0\}_{l\in I}$ is a family of non-negative cuts for P that satisfies Assumption 3.1, and suppose $V^c = V^c(\delta^l,\delta^l_0)$ for all $l \in I$. Then the set X is a polyhedron.*

We will prove Theorem 3.1 by induction on $|V \setminus V^c| + |E|$.

3.3.1 The basic step of the induction We first consider the case when $|V \setminus V^c| + |E| = 1$. The proof of Theorem 3.1 in this special case is by induction on $|V^c|$, and this proof is essentially the same for both the case when $|V \setminus V^c| = 1$, and the case when $|E| = 1$. We therefore assume $E = \{1\}$ and $|V \setminus V^c| = 0$ in the remainder of this subsection. We first consider the case when $|V^c| = 1$.

LEMMA 3.6 *(Lemma 7 in [1]). Suppose $|V^c| = 1$, $|V \setminus V^c| = 0$ and $E = \{1\}$. Then there exists $\bar{l} \in I$ such that $X = \{x \in P : (\delta^{\bar{l}})^T x \geq \delta_0^{\bar{l}}\}$.*

PROOF. For simplicity assume $V^c = \{1\}$. We have

$$P = \{x \in \mathbb{R}^n : x = v^1 + \mu_1 r^1 \text{ and } \mu_1 \geq 0\}, \text{ and}$$
$$\{x \in P : (\delta^l)^T x \geq \delta_0^l\} = \{x \in \mathbb{R}^n : x = v^1 + \mu_1 r^1, \mu_1 \geq 0 \text{ and } \frac{\mu_1}{\alpha'_{1,1}(\delta^l, \delta_0^l)} \geq 1\}$$

for all $l \in I$. Defining $\alpha^*_{1,1} := \sup\{\alpha'_{1,1,l} : l \in I\}$ then gives

$$X = \{x \in \mathbb{R}^n : x = v^1 + \mu_1 r^1, \mu_1 \geq 0 \text{ and } \frac{\mu_1}{\alpha^*_{1,1}} \geq 1\}.$$

Hence the only issue that needs to be verified is that the value $\alpha^*_{1,1}$ is attained for some $l \in I$. If there exists $l \in I$ satisfying $(\delta^l)^T r^1 = 0$, we are done, so we may assume $\alpha'_{1,1,l} < +\infty$ for all $l \in I$. Choosing $l' \in I$ arbitrarily, Assumption 3.1.(i) shows the set $\mathrm{IP}^e_{(1,1)}(\alpha_{1,1,l'})$ is finite. Therefore the supremum is achieved. □

The induction hypothesis is as follows. For every $i \in V^c$, define

$$P^i := \mathrm{conv}(\{v^{i'}\}_{i' \in V^c \setminus \{i\}}) + \mathrm{cone}(\{r^1\}), \text{ and}$$
$$X^i := \{x \in P^i : (\delta^l)^T x \geq \delta_0^l \text{ for all } l \in I\}.$$

The induction hypothesis is that X^i is a polyhedron for all $i \in V^c$. Hence, for every $i \in V^c$, we can choose a *finite* subset $I^i \subseteq I$ such that $X^i = \{x \in P^i : (\delta^l)^T x \geq \delta_0^l \text{ for all } l \in I^i\}$.

Let $\bar{I} := \cup_{i \in V^c} I^i$ denote the set of *all* inequalities needed to describe the sets X^i for $i \in V^c$. Also let $\overline{X} := \{x \in P : (\delta^l)^T x \geq \delta_0^l \text{ for } l \in \bar{I}\}$ be the approximation of X obtained from the finite set of inequalities indexed by $\bar{I}$, and define the numbers

$$\alpha^*_i := \min\{\alpha'_{i,1,l} : l \in \bar{I}\} > 0 \text{ for all } i \in V^c.$$

The number α^*_i gives the intersection point $v^i + \alpha'_{i,1,l} r^1$ which is closest to v^i over all inequalities $l \in \bar{I}$. Based on the induction hypothesis, we now show that X is a polyhedron when $|V \setminus V^c| = 0$ and $E = \{1\}$.

LEMMA 3.7 *(Lemma 8 in [1]). If $|V \setminus V^c| = 0$ and $E = \{1\}$, then X is a polyhedron.*

PROOF. Consider an inequality $l' \in I \setminus \bar{I}$. We will show that $(\delta^{l'})^T x \geq \delta_0^{l'}$ is valid for $\overline{X}$ if there exists $i' \in V^c$ such that $\alpha_{i',1,l'} \leq \alpha^*_{i'}$. This implies that it is sufficient to consider inequalities $l \in I \setminus \bar{I}$ that satisfy $\alpha_{i,1,l} > \alpha^*_i$ for all $i \in V^c$ to obtain X from $\overline{X}$. Since the sets $\mathrm{IP}^e_{(i,1)}(\alpha^*_i)$ for $i \in V^c$ are finite (Assumption 3.1.(i)), and since two inequalities with exactly the same intersection points are equivalent (Lemma 3.4), this shows that only a finite number of inequalities from $I \setminus \bar{I}$ are needed to obtain X from $\overline{X}$.

Therefore suppose $l' \in I \setminus \bar{I}$ and $i' \in V^c$ satisfies $\alpha_{i',1,l'} \leq \alpha^*_{i'}$. For simplicity let $(\delta', \delta'_0) := (\delta^{l'}, \delta_0^{l'})$. Since $(\delta')^T x \geq \delta'_0$ is a non-negative cut for $P^{i'}$ that is valid for $X^{i'}$ (the induction hypothesis), Lemma 3.5 shows there exists an inequality $\bar{\delta}^T x \geq \bar{\delta}_0$ that dominates $(\delta')^T x \geq \delta'_0$ on $P^{i'}$, and that $\bar{\delta}^T x \geq \bar{\delta}_0$ can be chosen as a convex combination of the inequalities $(\delta^l)^T x \geq \delta_0^l$ for $l \in \bar{I}$. We therefore have

$$\bar{\delta} = \sum_{l \in \bar{I}} \lambda_l \delta^l, \text{ and}$$
$$\bar{\delta}_0 = \sum_{l \in \bar{I}} \lambda_l \delta_0^l, \text{ where}$$
$$\sum_{l \in \bar{I}} \lambda_l = 1 \text{ and } \lambda_l \geq 0 \text{ for all } l \in \bar{I}.$$

We will show that $\bar{\delta}^T x \geq \bar{\delta}_0$ dominates $(\delta')^T x \geq \delta'_0$ on P by verifying that condition (i) of Lemma 3.4 is satisfied. We know $\bar{\delta}^T x \geq \bar{\delta}_0$ dominates $(\delta')^T x \geq \delta'_0$ on $P^{i'}$. Lemma 3.4 therefore gives

$$\frac{1}{\alpha'_{i,1}(\bar{\delta}, \bar{\delta}_0)} \leq \frac{1}{\alpha'_{i,1}(\delta', \delta'_0)} \text{ for all } i \in V^c \setminus \{i'\}.$$

To finish the proof, we will show $\frac{1}{\alpha'_{i',1}(\bar{\delta},\bar{\delta}_0)} \leq \frac{1}{\alpha'_{i',1}(\delta',\delta'_0)}$. The definition of $\alpha^*_{i'}$ gives

$$\alpha^*_{i'} \leq \alpha'_{i',1,l} = \frac{\delta^l_0 - (\delta^l)^T v^{i'}}{(\delta^l)^T r^1}$$

for all $l \in \bar{I}$. Since $\bar{\delta}_0 - \bar{\delta}^T v^{i'} = \sum_{l\in\bar{I}} \lambda_l \, (\delta^l_0 - (\delta^l)^T v^{i'})$ and $\bar{\delta}^T r^1 = \sum_{l\in\bar{I}} \lambda_l \, (\delta^l)^T r^1$, we obtain $\bar{\delta}_0 - \bar{\delta}^T v^{i'} \geq \alpha^*_{i'} \bar{\delta}^T r^1$, and therefore $\alpha^*_{i'} \leq \alpha'_{i',1}(\bar{\delta}, \bar{\delta}_0)$. The choice of $\alpha^*_{i'}$ and $(\delta')^T x \geq \delta'_0$ gives $\alpha_{i',1}(\delta', \delta'_0) \leq \alpha^*_{i'}$. Hence $\alpha'_{i',1}(\delta', \delta'_0) \leq \alpha'_{i',1}(\bar{\delta}, \bar{\delta}_0)$, which implies $1/\alpha'_{i',1}(\bar{\delta}, \bar{\delta}_0) \leq 1/\alpha'_{i',1}(\delta', \delta'_0)$. □

3.3.2 The induction hypothesis We now present the induction hypothesis. Given a vertex v^k of P with $k \in V \setminus V^c$, consider the polyhedron obtained from P by deleting v^k

$$P^k := \text{conv}(\{v^i\}_{i\in V\setminus\{k\}}) + \text{cone}(\{r^j\}_{j\in E}),$$

and given an extreme ray r^j of P with $j \in E$, consider the polyhedron obtained from P by deleting r^j

$$P^j := \text{conv}(\{v^i\}_{i\in V}) + \text{cone}(\{r^{j'}\}_{j'\in E\setminus\{j\}}).$$

From the inequalities $(\delta^l)^T x \geq \delta^l_0$ for $l \in I$, and the polyhedra P^k and P^j, we can define the following subsets of X.

$$X^k := \{x \in P^k : (\delta^l)^T x \geq \delta^l_0 \text{ for } l \in I\}, \text{ and}$$
$$X^j := \{x \in P^j : (\delta^l)^T x \geq \delta^l_0 \text{ for } l \in I\},$$

The induction hypothesis is that the sets X^k and X^j are polyhedra for all $k \in V \setminus V^c$ and $j \in E$. This implies that for every $k \in V \setminus V^c$, there exists a finite set $I^k \subseteq I$ such that

$$X^k = \{x \in P^k : (\delta^l)^T x \geq \delta^l_0 \text{ for } l \in I^k\},$$

and for every $j \in E$, there exists a finite set $I^j \subseteq I$ such that

$$X^j = \{x \in P^j : (\delta^l)^T x \geq \delta^l_0 \text{ for } l \in I^j\}.$$

Define $\bar{I} := (\cup_{k\in V\setminus V^c} I^k) \cup (\cup_{j\in E} I^j)$ to be the set of *all* inequalities involved above. The set $\bar{I}$ gives the following approximation $\overline{X}$ of X.

$$\overline{X} := \{x \in P : (\delta^l)^T x \geq \delta^l_0 \text{ for all } l \in \bar{I}\}$$

3.3.3 The inductive proof We now use the induction hypothesis to prove that X is a polyhedron. The idea of the proof is based on counting the number $|\text{SIP}(I')|$ of intersection points that are shared by *all* cuts in a family $I' \subseteq I$ of cutting planes. This number is given by $|\text{SIP}(I')| = |\text{SIP}^e(I')| + |\text{SIP}^v(I')|$, where the sets $\text{SIP}^e(I')$ and $\text{SIP}^v(I')$ are defined by

$$\text{SIP}^e(I') := \{(i,j) \in V^c \times E : \alpha'_{i,j,l_1} = \alpha'_{i,j,l_2} \text{ for all } l_1, l_2 \in I'\}, \text{ and}$$
$$\text{SIP}^v(I') := \{(i,k) \in V^c \times (V \setminus V^c) : \beta'_{i,k,l_1} = \beta'_{i,k,l_2} \text{ for all } l_1, l_2 \in I'\}.$$

Clearly we have $0 \leq |\text{SIP}(I')| \leq |V^c \times E| + |V^c \times (V \setminus V^c)|$ for all $I' \subseteq I$. Furthermore, if $|\text{SIP}(I')| = |V^c \times E| + |V^c \times (V \setminus V^c)|$, then *all* cuts indexed by I' share *all* intersection points with the halflines $\{v^i + \alpha r^j : \alpha \geq 0\}$ and $\{v^i + \beta(v^k - v^i) : \beta \geq 0\}$ for $i \in V^c$, $j \in E$ and $k \in V \setminus V^c$. This then implies that all cuts indexed by I' are equivalent on P (Lemma 3.4). Therefore, if $|\text{SIP}(I')| = |V^c \times E| + |V^c \times (V \setminus V^c)|$, then the set X' given by $X' := \{x \in P : (\delta^l)^T x \geq \delta^l_0 \text{ for all } l \in I'\}$ is a polyhedron that can be described with exactly one cut from the family I'.

The main idea of our proof can now be presented. Clearly we can assume that the family $\bar{I}$ does not give a complete description of X (otherwise there is nothing to prove). We will show the following lemma.

LEMMA 3.8 *Assume the sets $\{X^k\}_{k\in V\setminus V^c}$ and $\{X^j\}_{j\in E}$ are polyhedra. There exists a covering of I into a finite number of subsets $\{I^q\}_{q=1}^{ns}$ such that*

$$\textit{for all } q \in \{1,2,\ldots,ns\}, \textit{ either } I^q \subseteq \bar{I}, \textit{ or } |SIP(I^q)| > |SIP(I)|,$$

where ns denotes the number of subsets in this covering.

The goal of the remainder of this section is to prove Lemma 3.8. We first argue that Lemma 3.8 implies that X is a polyhedron. The fact that $\{I^q\}_{q=1}^{\mathrm{ns}}$ is a covering of I implies

$$X = \cap_{q=1}^{\mathrm{ns}} X(I^q),$$

where $X(I^q) := \{x \in P : (\delta^l)^T x \geq \delta_0^l \text{ for all } l \in I^q\}$. Therefore X is a polyhedron if $X(I^q)$ is a polyhedron for all $q \in \{1, 2, \ldots, \mathrm{ns}\}$. Since $|\mathrm{SIP}(I^q)|$ is larger than $|\mathrm{SIP}(I)|$ for all $q \in \{1, 2, \ldots, \mathrm{ns}\}$ satisfying $I^q \nsubseteq \bar{I}$, recursively applying Lemma 3.8 will create a tree of subcases, where the sets corresponding to the leaves of this tree must be polyhedra. It then follows that X is a polyhedron.

We now proceed to prove Lemma 3.8. The covering of I is based on the following positive numbers that measure how close the cuts $\{(\delta^l)^T x \geq \delta_0^l\}_{l \in \bar{I}}$ cut to a vertex v^i of P.

$$\alpha_j^* := \min\{\alpha'_{i,j,l} : i \in V^c \text{ and } l \in \bar{I}\} \text{ for } j \in E, \text{ and}$$
$$\beta_k^* := \min\{\beta'_{i,k,l} : i \in V^c \text{ and } l \in \bar{I}\} \text{ for } k \in V \setminus V^c.$$

Given $\bar{j} \in E$, the number $\alpha_{\bar{j}}^*$ corresponds to a vertex $v^{\bar{i}}$ of P and a cut $(\delta^{\bar{l}})^T x \geq \delta_0^{\bar{l}}$ for which the intersection point $v^{\bar{i}} + \alpha'_{\bar{i},\bar{j},\bar{l}} r^{\bar{j}}$ is as close to $v^{\bar{i}}$ as possible. Similarly, given $\bar{k} \in V \setminus V^c$, the number $\beta_{\bar{k}}^*$ corresponds to a vertex $v^{\bar{i}}$ of P and a cut $(\delta^{\bar{l}})^T x \geq \delta_0^{\bar{l}}$ for which the intersection point $v^{\bar{i}} + \beta'_{\bar{i},\bar{k},\bar{l}}(v^{\bar{k}} - v^{\bar{i}})$ is as close to $v^{\bar{i}}$ as possible.

The numbers $\{\alpha_j^*\}_{j \in E}$ and $\{\beta_k^*\}_{k \in V \setminus V^c}$ allow us to provide the following conditon that the cuts $(\delta^l)^T x \geq \delta_0^l$ for $l \in I \setminus \bar{I}$ must satisfy in order to cut off a region of $\overline{X}$. Clearly cuts that are valid for $\overline{X}$ can be removed from $I \setminus \bar{I}$, since they do not contribute anything further to the description of X than the cuts indexed by $\bar{I}$.

Lemma 3.9 *(Lemma 8 in [1]). Assume the sets $\{X^k\}_{k \in V \setminus V^c}$ and $\{X^j\}_{j \in E}$ are polyhedra, and let $\bar{l} \in I \setminus \bar{I}$ be arbitrary. If either*

(i) There exists $\bar{j} \in E$ such that $\max\{\alpha'_{i,\bar{j},\bar{l}} : i \in V^c$ and $(i, \bar{j}) \notin SIP^e(I)\} \leq \alpha_{\bar{j}}^*$
(The cut $(\delta^{\bar{l}})^T x \geq \delta_0^{\bar{l}}$ does not cut off any point of the form $v^i + \alpha_{\bar{j}}^* r^{\bar{j}}$ which is not an intersection point that is shared by all cuts in I), or

(ii) There exists $\bar{k} \in V \setminus V^c$ such that $\max\{\beta'_{i,\bar{k},\bar{l}} : i \in V^c$ and $(i, \bar{k}) \notin SIP^v(I)\} \leq \beta_{\bar{k}}^*$
(The cut $(\delta^{\bar{l}})^T x \geq \delta_0^{\bar{l}}$ does not cut off any point of the form $v^i + \beta_{\bar{k}}^*(v^{\bar{k}} - v^i)$ which is not an intersection point that is shared by all cuts in I),

then the cut $(\delta^{\bar{l}})^T x \geq \delta_0^{\bar{l}}$ is valid for $\overline{X}$.

The proof of Lemma 3.9 will be given at the end of this section. We first argue that Lemma 3.9 can be used to prove Lemma 3.8, which thereby finishes the proof of Theorem 3.1.

Lemma 3.9 shows we can partition the cuts $\{(\delta^l)^T x \geq \delta_0^l\}_{l \in I}$ into three categories.

(1) The cuts indexed by $\bar{I}$ that define $\bar{X}$.
(2) The cuts, indexed by some set $I^r \subseteq I \setminus \bar{I}$, that satisfy either Lemma 3.9.(i) or Lemma 3.9.(ii), and these cuts are valid for $\overline{X}$.
(3) The remainder of the cuts indexed by $I \setminus (\bar{I} \cup I^r)$. Every cut $l \in I \setminus (\bar{I} \cup I^r)$ satisfies:
   (i) For all $j \in E$, the cut $(\delta^l)^T x \geq \delta_0^l$ cuts off some intersection point of the form $v^i + \alpha_j^* r^{\bar{j}}$, which is not an intersection point that is shared by all cuts in I.
   (ii) For all $k \in V \setminus V^c$, the cut $(\delta^l)^T x \geq \delta_0^l$ cuts off some intersection point of the form $v^i + \beta_k^*(v^k - v^i)$, which is not an intersection point that is shared by all cuts in I.

Clearly we can assume $I^r = \emptyset$. Let $i \in V^c$, $j \in E$ and $k \in V \setminus V^c$ be arbitrary. Recall that the sets $\mathrm{IP}^e_{(i,j)}(\alpha_j^*)$ and $\mathrm{IP}^v_{(i,k)}(\beta_k^*)$ identify the intersection points between the hyperplanes $\{(\delta^l)^T x = \delta_0^l\}_{l \in I}$ and the halflines $\{v^i + \alpha r^j : \alpha \geq \alpha_j^*\}$ and $\{v^i + \beta(v^k - v^i) : \beta \geq \beta_k^*\}$ respectively. Hence we may write $\mathrm{IP}^e_{(i,j)}(\alpha_j^*)$ and $\mathrm{IP}^v_{(i,k)}(\beta_k^*)$ in the form

$$\mathrm{IP}^e_{(i,j)}(\alpha_j^*) = \{\alpha_{i,j}^1, \alpha_{i,j}^2, \ldots, \alpha_{i,j}^{n^e(i,j)}\} \text{ and}$$
$$\mathrm{IP}^v_{(i,k)}(\beta_k^*) = \{\beta_{i,k}^1, \beta_{i,k}^2, \ldots, \beta_{i,k}^{n^v(i,k)}\},$$

where the numbers $n^e(i,j) := |\mathrm{IP}^e_{(i,j)}(\alpha^*_j)|$ and $n^v(i,k) := |\mathrm{IP}^v_{(i,k)}(\beta^*_k)|$ denote the sizes of the two sets. For simplicity let $N^e_{(i,j)} := \{1, 2, \dots, n^e(i,j)\}$ and $N^v_{(i,k)} := \{1, 2, \dots, n^v(i,k)\}$ index the intersection points.

All intersection points between a hyperplane $(\delta^l)^T x = \delta^l_0$ with $l \in I \setminus \bar{I}$ and a halfline either of the form $\{v^i + \alpha r^j : \alpha \geq \alpha^*_j\}$, or of the form $\{v^i + \beta(v^k - v^i) : \beta \geq \beta^*_k\}$, can be identified with elements of the index sets

$$\mathrm{AIP}^e := \{(i,j,q) : i \in V^c, j \in E \text{ and } q \in N^e_{(i,j)}\}, \text{ and}$$
$$\mathrm{AIP}^v := \{(i,k,q) : i \in V^c, k \in V \setminus V^c \text{ and } q \in N^v_{(i,k)}\}.$$

For a specific cut $(\delta^l)^T x \geq \delta^l_0$ with $l \in I$, let the sets

$$\mathrm{IP}^e(l) := \{(i,j,q) \in \mathrm{AIP}^e : \alpha'_{i,j,l} = \alpha^q_{i,j}\}$$
$$\mathrm{IP}^v(l) := \{(i,k,q) \in \mathrm{AIP}^v : \beta'_{i,k,l} = \beta^q_{i,k}\}.$$

index the intersection points between $(\delta^l)^T x = \delta^l_0$ and the halflines $\{v^i + \alpha r^j : \alpha \geq \alpha^*_j\}$ and $\{v^i + \beta(v^k - v^i) : \beta \geq \beta^*_k\}$ for $i \in V^c$, $j \in E$ and $k \in V \setminus V^c$.

Observe that, from the definitions of α^*_j and β^*_k for $j \in E$ and $k \in V \setminus V^c$, we have $\mathrm{IP}^e(l) \neq \emptyset$ and $\mathrm{IP}^v(l) \neq \emptyset$ for all $l \in \bar{I}$. Furthermore, property (3) above and the assumption $I^r = \emptyset$ ensures that $\mathrm{IP}^e(l) \neq \emptyset$ and $\mathrm{IP}^v(l) \neq \emptyset$ for all $l \in I \setminus \bar{I}$. Hence we have $\mathrm{IP}^e(l) \neq \emptyset$ and $\mathrm{IP}^v(l) \neq \emptyset$ for all $l \in I$.

Given a pair $(S^e, S^v) \subseteq \mathrm{AIP}^e \times \mathrm{AIP}^v$, the sets S^e and S^v may or may not denote the index sets for all intersection points between a specific hyperplane $(\delta^l)^T x = \delta^l_0$ and the halflines $\{v^i + \alpha r^j : \alpha \geq \alpha^*_j\}$ and $\{v^i + \beta(v^k - v^i) : \beta \geq \beta^*_k\}$ for $i \in V^c$, $j \in E$, $k \in V \setminus V^c$ and $l \in I$. Let

$$\mathcal{S}^* := \{(S^e, S^v) \subseteq \mathrm{AIP}^e \times \mathrm{AIP}^v : S^e = \mathrm{IP}^e(l) \text{ and } S^v = \mathrm{IP}^v(l) \text{ for some } l \in I\}$$

denote the set of *all* pairs (S^e, S^v) that describe the index sets for the intersection points for *some* cutting plane $l \in I$. For a given pair $(S^e, S^v) \in \mathcal{S}^*$, let

$$\mathrm{CA}(S^e, S^v) := \{l \in I : S^e = \mathrm{IP}^e(l) \text{ and } S^v = \mathrm{IP}^e(l)\}$$

denote the set of *all* cuts associated with the pair (S^e, S^v), *i.e.*, the set of all cuts whose intersection points with the halflines $\{v^i + \alpha r^j : \alpha \geq \alpha^*_j\}$ and $\{v^i + \beta(v^k - v^i) : \beta \geq \beta^*_k\}$ for $i \in V^c$, $j \in E$ and $k \in V \setminus V^c$ are characterized by the pair (S^e, S^v).

We claim that the finite number of sets $\{\mathrm{CA}(S^e, S^v)\}_{(S^e,S^v)\in\mathcal{S}^*}$ provides the covering of I that is claimed to exist in Lemma 3.8. Indeed, the fact that $\mathrm{IP}^e(l) \neq \emptyset$ and $\mathrm{IP}^v(l) \neq \emptyset$ for all $l \in I$ implies that every cut $l \in I$ belongs to *some* set $\mathrm{CA}(S^e, S^v)$ with $(S^e, S^v) \in \mathcal{S}^*$. Hence $\{\mathrm{CA}(S^e, S^v)\}_{(S^e,S^v)\in\mathcal{S}^*}$ is a covering of I.

Let $(S^e, S^v) \in \mathcal{S}^*$ be arbitrary. If $\mathrm{CA}(S^e, S^v) \subseteq \bar{I}$, then clearly the condition in Lemma 3.8 is satisfied for (S^e, S^v), so we may assume $\mathrm{CA}(S^e, S^v)$ contains cuts from $I \setminus \bar{I}$. Furthermore, we clearly have $\mathrm{SIP}(I) \subseteq \mathrm{SIP}(\mathrm{CA}(S^e, S^v))$, since $\mathrm{CA}(S^e, S^v)$ is a subset of I. To finish the proof of Lemma 3.8, we need to show that $|\mathrm{SIP}(\mathrm{CA}(S^e, S^v))| > |\mathrm{SIP}(I)|$.

Lemma 3.9.(i) shows that for every $l \in I \setminus \bar{I}$ and $j \in E$, there exists $i \in V^c$ such that $(i,j) \notin \mathrm{SIP}^e(I)$ and $\alpha'_{i,j}(\delta^l, \delta^l_0) > \alpha^*_j$. Furthermore, 3.9.(ii) shows that for every $l \in I \setminus \bar{I}$ and $k \in V \setminus V^c$, there exists $i \in V^c$ such that $(i,k) \notin \mathrm{SIP}^v(I)$ and $\beta'_{i,k}(\delta^l, \delta^l_0) > \beta^*_k$. This shows the existence of a cut $\bar{l} \in \mathrm{CA}(S^e, S^v)$ that satisfies $\bar{l} \notin \mathrm{SIP}(I)$, and therefore $|\mathrm{SIP}(\mathrm{CA}(S^e, S^v))| > |\mathrm{SIP}(I)|$. This completes the proof of Theorem 3.1.

Proof of Lemma 3.9. The proof of (ii) is the same as the proof of (i), so we only show (i). Therefore suppose the cut $\bar{l} \in I \setminus \bar{I}$ and the extreme ray $\bar{j} \in E$ satisfies the inequality $\max\{\alpha'_{i,\bar{j},\bar{l}} : i \in V^c \text{ and } (i,\bar{j}) \notin \mathrm{SIP}^e(I)\} \leq \alpha^*_{\bar{j}}$. For simplicity let $(\delta', \delta'_0) := (\delta^{l'}, \delta^{l'}_0)$.

Since $(\delta')^T x \geq \delta'_0$ is a non-negative cut for $P^{\bar{j}}$, Lemma 3.5 shows there exists an inequality $\bar{\delta}^T x \geq \bar{\delta}_0$ that dominates $(\delta')^T x \geq \delta'_0$ on $P^{\bar{j}}$, and that this inequality can be chosen to be a convex combination of

the inequalities $(\delta^l)^T x \geq \delta^l_0$ for $l \in \bar{I}$. Hence

$$\bar{\delta} = \sum_{l\in\bar{I}} \lambda_l \delta^l,$$
$$\bar{\delta}_0 = \sum_{l\in\bar{I}} \lambda_l \delta^l_0, \text{ where}$$
$$\sum_{l\in\bar{I}} \lambda_l = 1 \text{ and } \lambda_l \geq 0 \text{ for all } l \in \bar{I}.$$

We will show that $\bar{\delta}^T x \geq \bar{\delta}_0$ also dominates $(\delta')^T x \geq \delta'_0$ on P by verifying that conditions (i) and (ii) of Lemma 3.4 are satisfied. Since $\bar{\delta}^T x \geq \bar{\delta}_0$ dominates $(\delta')^T x \geq \delta'_0$ on $P^{\bar{j}}$, we have

$$\frac{1}{\alpha'_{i,j}(\bar{\delta},\bar{\delta}_0)} \leq \frac{1}{\alpha'_{i,j}(\delta',\delta'_0)} \text{ for all } i \in V^c \text{ and } j \in E \setminus \{\bar{j}\}, \text{ and}$$
$$\frac{1}{\beta'_{i,k}(\bar{\delta},\bar{\delta}_0)} \leq \frac{1}{\alpha'_{i,k}(\delta',\delta'_0)} \text{ for all } i \in V^c \text{ and } k \in V \setminus V^c.$$

We also know

$$\frac{1}{\alpha'_{i,\bar{j}}(\bar{\delta},\bar{\delta}_0)} = \frac{1}{\alpha'_{i,\bar{j}}(\delta',\delta'_0)} \text{ for all } i \in V^c \text{ such that } (i,\bar{j}) \in \mathrm{SIP}^e(I).$$

To finish the proof, it suffices to show

$$\frac{1}{\alpha'_{i,\bar{j}}(\bar{\delta},\bar{\delta}_0)} \leq \frac{1}{\alpha'_{i,\bar{j}}(\delta',\delta'_0)} \text{ for all } i \in V^c \text{ such that } (i,\bar{j}) \notin \mathrm{SIP}^e(I).$$

From the definition of $\alpha^*_{\bar{j}}$, we have the inequality

$$\alpha^*_{\bar{j}} \leq \alpha'_{i,\bar{j},l} = \frac{\delta^l_0 - (\delta^l)^T v^i}{(\delta^l)^T r^{\bar{j}}}$$

for all $l \in \bar{I}$ and $i \in V^c$. The equalities $\bar{\delta}_0 - \bar{\delta}^T v^i = \sum_{l\in\bar{I}} \lambda_l\ (\delta^l_0 - (\delta^l)^T v^i)$ for all $i \in V^c$, and $\bar{\delta}^T r^{\bar{j}} = \sum_{l\in\bar{I}} \lambda_l\ (\delta^l)^T r^{\bar{j}}$ imply $\bar{\delta}_0 - \bar{\delta}^T v^i \geq \alpha^*_{\bar{j}} \bar{\delta}^T r^{\bar{j}}$ for all $i \in V^c$, and therefore $\alpha^*_{\bar{j}} \leq \alpha'_{i,\bar{j}}(\bar{\delta},\bar{\delta}_0)$ for all $i \in V^c$. The definition of $\alpha^*_{\bar{j}}$ and the choice of the cut $(\delta')^T x \geq \delta'_0$ imply $\alpha'_{\bar{i},\bar{j}}(\delta',\delta'_0) \leq \max\{\alpha'_{i,\bar{j}}(\delta',\delta'_0) : i \in V^c \text{ such that } (i,\bar{j}) \notin \mathrm{SIP}^e(I)\} \leq \alpha^*_{\bar{j}}$ for all $\bar{i} \in V^c$ such that $(\bar{i},\bar{j}) \notin \mathrm{SIP}^e(I)$, and therefore $\alpha'_{i,\bar{j}}(\delta',\delta'_0) \leq \alpha'_{i,\bar{j}}(\bar{\delta},\bar{\delta}_0)$ for all $i \in V^c$ such that $(i,\bar{j}) \notin \mathrm{SIP}^e(I)$. Hence $1/\alpha'_{i,\bar{j}}(\bar{\delta},\bar{\delta}_0) \leq 1/\alpha'_{i,\bar{j}}(\delta',\delta'_0)$ for all $i \in V^c$ satisfying $(i,\bar{j}) \notin \mathrm{SIP}^e(I)$, which gives that $\bar{\delta}^T x \geq \bar{\delta}_0$ dominates $(\delta')^T x \geq \delta'_0$ on P. □

4. The structure of polyhedral relaxations obtained from mixed integer split polyhedra We now describe the polyhedral structure of the polyhedron $R(L,P)$ for a mixed integer split polyhedron L. Throughout this section, L denotes an arbitrary mixed integer split polyhedron. Also, $V^{\text{in}}(L) := \{i \in V : v^i \in \text{int}(L)\}$ denotes the vertices of P in the interior of L and $V^{\text{out}}(L) := V \setminus V^{\text{in}}(L)$ denotes the vertices of P that are *not* in the interior of L. We assume $V^{\text{in}}(L) \neq \emptyset$, since otherwise $R(L,P) = P$ (Lemma 2.5). The set $\Lambda := \{\lambda \in \mathbb{R}^{|V|} : \lambda \geq 0 \text{ and } \sum_{i\in V} \lambda_i = 1\}$ is used to form convex combinations of the vertices of P, and the set $\Lambda^{\text{in}}(L) := \{\lambda \in \Lambda : \sum_{i\in V^{\text{in}}(L)} \lambda_i = 1\}$ is used to form convex combinations of the vertices in $V^{\text{in}}(L)$.

4.1 Intersection points Now consider possible intersection points between a halfline of the form $\{v_{\lambda^{\text{in}}} + \alpha r^j : \alpha \geq 0\}$ and the boundary of L, where $\lambda^{\text{in}} \in \Lambda^{\text{in}}(L)$ and $j \in E$. Given $\lambda^{\text{in}} \in \Lambda^{\text{in}}(L)$ and $j \in E$, define:

$$\alpha_j(L,\lambda^{\text{in}}) := \sup\{\alpha : v_{\lambda^{\text{in}}} + \alpha r^j \in L\}. \tag{9}$$

The number $\alpha_j(L,\lambda^{\text{in}}) > 0$ determines the closest point $v_{\lambda^{\text{in}}} + \alpha_j(L,\lambda^{\text{in}}) r^j$ (if any) to $v_{\lambda^{\text{in}}}$ on the halfline $\{v_{\lambda^{\text{in}}} + \alpha r^j : \alpha \geq 0\}$ which is *not* in the interior of L. Observe that if $\{v_{\lambda^{\text{in}}} + \alpha r^j : \alpha \geq 0\} \subseteq \text{int}(L)$, then $\alpha_j(L,\lambda^{\text{in}}) = +\infty$. When $\alpha_j(L,\lambda^{\text{in}}) < +\infty$, the point $v_{\lambda^{\text{in}}} + \alpha_j(L,\lambda^{\text{in}}) r^j$ is called an *intersection point*.

The value $\alpha_j(L,\lambda^{\text{in}})$ is a function of λ^{in}. This function has the following important property. Given any convex set $C \subseteq \mathbb{R}^{n+1}$, it is well known (see Rockafellar [8]) that the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) := \sup\{\mu : (x,\mu) \in C\}$$

is a concave function. Now, given any $\lambda^{\text{in}} \in \Lambda^{\text{in}}(L)$ and $j \in E$, we may write

$$\alpha_j(L, \lambda^{\text{in}}) = \sup\{\alpha : (\lambda^{\text{in}}, \alpha) \in \tilde{P}(L)\},$$

where $\tilde{P}(L)$ is the convex polyhedron $\tilde{P}(L) := \{(\lambda^{\text{in}}, \alpha) \in \mathbb{R}^{|V^{\text{in}}(L)|+1} : v_{\lambda^{\text{in}}} + \alpha r^j \in L\}$. We therefore have that the function $\alpha_j(L, \lambda^{\text{in}})$ has the following property.

Lemma 4.1 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$, and let $j \in E$. The function $\alpha_j(L, \lambda^{in})$ is concave in λ^{in}, i.e., for every $\lambda^1, \lambda^2 \in \Lambda^{in}(L)$ and $\mu \in [0, 1]$, we have $\alpha_j(L, \mu\lambda^1 + (1-\mu)\lambda^2) \geq \mu\alpha_j(L, \lambda^1) + (1-\mu)\alpha_j(L, \lambda^2)$.*

Given a convex combination $\lambda^{\text{in}} \in \Lambda^{\text{in}}(L)$, and a vertex $k \in V^{\text{out}}(L)$, the line between $v_{\lambda^{\text{in}}}$ and v^k intersects the boundary of L. For $k \in V^{\text{out}}(L)$ and $\lambda^{\text{in}} \in \Lambda^{\text{in}}(L)$, define

$$\beta_k(L, \lambda^{\text{in}}) := \sup\{\beta : v_{\lambda^{\text{in}}} + \beta(v^k - v_{\lambda^{\text{in}}}) \in L\}. \tag{10}$$

The number $\beta_k(L, \lambda^{\text{in}})$ denotes the value of β for which the point $v_{\lambda^{\text{in}}} + \beta(v^k - v_{\lambda^{\text{in}}})$ is on the boundary of L. The point $v_{\lambda^{\text{in}}} + \beta(v^k - v_{\lambda^{\text{in}}})$ is also called an intersection point, and we observe that $\beta_k(L, \lambda^{\text{in}}) \in]0, 1]$. The intersection point $v_{\lambda^{\text{in}}} + \beta_k(L, \lambda^{\text{in}})(v^k - v_{\lambda^{\text{in}}})$ has the following important property.

Lemma 4.2 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$, and let $k \in V^{out}(L)$. For every $\lambda^{in} \in \Lambda^{in}(L)$, the intersection point $v_{\lambda^{in}} + \beta_k(L, \lambda^{in})(v^k - v_{\lambda^{in}})$ is a convex combination of v^k and the intersection points $v^i + \beta_k(L, e^i)(v^k - v^i)$ for $i \in V^{in}(L)$.*

Proof. Define $C := \text{conv}(\{v^i + \beta_k(L, e^i)(v^k - v^i)\}_{i \in V^{\text{in}}(L)})$. We first show that the halfline $\{v_{\lambda^{\text{in}}} + \beta(v^k - v_{\lambda^{\text{in}}}) : \beta \geq 0\}$ intersects C for some $\beta^* > 0$. We have that $\{v_{\lambda^{\text{in}}} + \beta(v^k - v_{\lambda^{\text{in}}}) : \beta \geq 0\} \cap C \neq \emptyset$ if and only if the following LP is feasible.

$$\begin{aligned}
\min\ & 0 \\
& \sum_{i \in V^{\text{in}}(L)} \eta_i(v^i + \beta_k(L, e^i)(v^k - v^i)) + \beta(v_{\lambda^{\text{in}}} - v^k) = v_{\lambda^{\text{in}}}, && (11)\\
& \sum_{i \in V^{\text{in}}(L)} \eta_i = 1, && (12)\\
& \eta, \beta \geq 0. && (13)
\end{aligned}$$

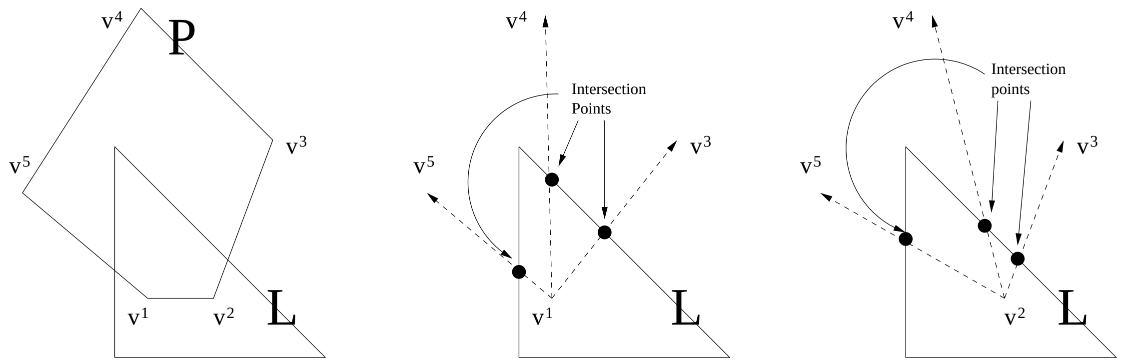
The dual of this LP is given by

$$\begin{aligned}
\max\ & \delta^T v_{\lambda^{\text{in}}} - \delta_0 \\
& \delta^T(v_{\lambda^{\text{in}}} - v^k) \leq 0, && (14)\\
& \delta^T(v^i + \beta_k(L, e^i)(v^k - v^i)) - \delta_0 \leq 0, \text{ for all } i \in V^{\text{in}}(L). && (15)
\end{aligned}$$

Let $(\bar{\delta}, \bar{\delta}_0)$ be a solution to (14)-(15). Suppose, for a contradiction, that $\bar{\delta}^T v_{\lambda^{\text{in}}} - \bar{\delta}_0 > 0$. Adding (14) to the inequality of (15) corresponding to $\bar{i} \in V^{\text{in}}(L)$ gives $\bar{\delta}^T v_{\lambda^{\text{in}}} - \bar{\delta}_0 + (1 - \beta_k(L, e^{\bar{i}}))\bar{\delta}^T(v^{\bar{i}} - v^k) \leq 0$. Since by assumption $\bar{\delta}^T v_{\lambda^{\text{in}}} - \bar{\delta}_0 > 0$, this implies $\bar{\delta}^T(v^{\bar{i}} - v^k) < 0$. Hence we have $\bar{\delta}^T(v^i - v^k) < 0$ for all $i \in V^{\text{in}}(L)$. Now, for all $i \in V^{\text{in}}(L)$, inequality (15) gives $\bar{\delta}_0 - \bar{\delta}^T v^i \geq \beta_k(L, e^i)\bar{\delta}^T(v^k - v^i)$. Since $\bar{\delta}^T(v^k - v^i) > 0$ for all $i \in V^{\text{in}}(L)$, this implies $\bar{\delta}_0 - \bar{\delta}^T v^i > 0$ for all $i \in V^{\text{in}}(L)$. Multiplying each of the inequalities $\bar{\delta}_0 - \bar{\delta}^T v^i > 0$ for $i \in V^{\text{in}}(L)$ with λ_i^{in} and adding the resulting inequalities together then gives $\bar{\delta}_0 - \bar{\delta}^T v_{\lambda^{\text{in}}} > 0$. This contradicts our initial assumption that $\bar{\delta}^T v_{\lambda^{\text{in}}} - \bar{\delta}_0 > 0$.

Therefore there exists $\beta^* \geq 0$ s.t. $v_{\lambda^{\text{in}}} + \beta^*(v^k - v_{\lambda^{\text{in}}}) \in C$. Observe that, since $v^i + \beta_k(L, e^i)(v^k - v^i) \in L$ for all $i \in V^{\text{in}}(L)$, we have $v_{\lambda^{\text{in}}} + \beta^*(v^k - v_{\lambda^{\text{in}}}) \in L$. If $v_{\lambda^{\text{in}}} + \beta^*(v^k - v_{\lambda^{\text{in}}}) \in \text{int}(L)$, then $\beta_k(L, \lambda^{\text{in}}) > \beta^*$, and therefore $v_{\lambda^{\text{in}}} + \beta_k(L, \lambda^{\text{in}})(v^k - v_{\lambda^{\text{in}}}) \in \text{conv}(C \cup \{v^k\})$. If $v_{\lambda^{\text{in}}} + \beta^*(v^k - v_{\lambda^{\text{in}}})$ is on the boundary of L, then $\beta_k(L, \lambda^{\text{in}}) = \beta^*$, which implies $v_{\lambda^{\text{in}}} + \beta_k(L, \lambda^{\text{in}})(v^k - v_{\lambda^{\text{in}}}) \in \text{conv}(C \cup \{v^k\})$. □

Lemma 4.2 shows that the only intersection points of the form $v_{\lambda^{\text{in}}} + \beta_k(L, \lambda^{\text{in}})(v^k - v_{\lambda^{\text{in}}})$ that can be vertices of $R(L, P)$ are those where λ^{in} is a unit vector. Figure 3 gives all the intersection points which can potentially be vertices of $R(L, P)$ for the example of Figure 1.



(a) The polytope P and the split polyhedron L from Figure 1

(b) Intersection points from v^1

(c) Intersection points from v^2

Figure 3: Determining the intersection points from a linear relaxation P and a split polyhedron L

4.2 The intersection cut In [3], Balas considered a mixed integer set defined from the translate of a polyhedral cone, and a mixed integer split polyhedron was used to derive a valid inequality for this set called the *intersection cut*. We now consider a subset $P(\lambda^{\rm in})$ of P defined from a fixed convex combination $\lambda^{\rm in} \in \Lambda^{\rm in}(L)$ of the vertices in the interior of L, and we show that the intersection cut gives a complete description of the set $R(L, P(\lambda^{\rm in}))$ in a higher dimensional space. Specifically, given any fixed convex combination $\lambda^{\rm in} \in \Lambda^{\rm in}(L)$, we have the following subset $P(\lambda^{\rm in})$ of P

$$P(\lambda^{\rm in}) = \{x \in \mathbb{R}^n : x = v_{\lambda^{\rm in}} + \sum_{k \in V^{\rm out}(L)} \epsilon_k (v^k - v_{\lambda^{\rm in}}) + \sum_{j \in E} \mu_j r^j, \mu \geq 0 \text{ and } \epsilon \in \Lambda^{\rm out}_{\leq}\},$$

where $\Lambda^{\rm out}_{\leq} := \{\lambda \in \mathbb{R}^{|V|} : \sum_{k \in V^{\rm out}} \lambda_k \leq 1 \text{ and } \lambda \geq 0\}$. The corresponding lifted image $P^l(\lambda^{\rm in})$ of $P(\lambda^{\rm in})$ in (x, ϵ, μ) space is given by

$$P^l(\lambda^{\rm in}) = \{(x, \epsilon, \mu) \in \mathbb{R}^{n+|V|+|E|} : x = v_{\lambda^{\rm in}} + \sum_{k \in V^{\rm out}(L)} \epsilon_k (v^k - v_{\lambda^{\rm in}}) + \sum_{j \in E} \mu_j r^j, \mu \geq 0 \text{ and } \epsilon \in \Lambda^{\rm out}_{\leq}\}.$$

The set $P(\lambda^{\rm in})$ and the mixed integer split polyhedron L gives a relaxation $R(L, P(\lambda^{\rm in}))$ of the set of mixed integer points in $P(\lambda^{\rm in})$

$$R(L, P(\lambda^{\rm in})) = \text{conv}(\{x \in P(\lambda^{\rm in}) : x \notin \text{int}(L)\}).$$

The lifted version $R^l(L, P(\lambda^{\rm in}))$ of $R(L, P(\lambda^{\rm in}))$ in (x, ϵ, μ) space is then defined to be the set $R^l(L, P(\lambda^{\rm in})) := \text{conv}(\{(x, \epsilon, \mu) \in P^l(\lambda^{\rm in}) : x \notin \text{int}(L)\})$. Given $\lambda^{\rm in} \in \Lambda^{\rm in}(L)$, and the corresponding intersection points, Balas [3] derived the *intersection cut*

$$\sum_{j \in E} \frac{\mu_j}{\alpha_j(L, \lambda^{\rm in})} + \sum_{k \in V^{\rm out}(L)} \frac{\epsilon_k}{\beta_k(L, \lambda^{\rm in})} \geq 1 \tag{16}$$

and showed that the intersection cut is valid for $R^l(L, P(\lambda^{\rm in}))$. We now show that, in fact, the intersection cut gives a complete description of $R^l(L, P(\lambda^{\rm in}))$.

THEOREM 4.1 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$, and let $\lambda^{in} \in \Lambda^{in}(L)$.*

$$R^l(L, P(\lambda^{in})) = \{(x, \epsilon, \mu) \in P^l(\lambda^{in}) : \sum_{j \in E} \frac{\mu_j}{\alpha_j(L, \lambda^{in})} + \sum_{k \in V^{out}(L)} \frac{\epsilon_k}{\beta_k(L, \lambda^{in})} \geq 1\}.$$

PROOF. Since (16) is valid for $R^l(L, P(\lambda^{\rm in}))$, we have

$$R^l(L, P(\lambda^{\rm in})) \subseteq \{(x, \epsilon, \mu) \in P^l(\lambda^{\rm in}) : \sum_{j \in E} \frac{\mu_j}{\alpha_j(L, \lambda^{\rm in})} + \sum_{k \in V^{\rm out}(L)} \frac{\epsilon_k}{\beta_k(L, \lambda^{\rm in})} \geq 1\}.$$

Conversely suppose $(\bar{x}, \bar{\epsilon}, \bar{\mu}) \in P^l(\lambda^{\rm in})$ and $\sum_{j \in E} \frac{\bar{\mu}_j}{\alpha_j(L, \lambda^{\rm in})} + \sum_{k \in V^{\rm out}(L)} \frac{\bar{\epsilon}_k}{\beta_k(L, \lambda^{\rm in})} \geq 1$. We will show that $(\bar{x}, \bar{\epsilon}, \bar{\mu}) \in R^l(L, P(\lambda^{\rm in}))$. Define $E^\infty := \{j \in E : \alpha_j(L, \lambda^{\rm in}) = +\infty\}$. We distinguish four cases.

(1) First suppose $\sum_{j\in E} \frac{\bar\mu_j}{\alpha_j(L,\lambda^{\text{in}})} + \sum_{k\in V^{\text{out}}(L)} \frac{\bar\epsilon_k}{\beta_k(L,\lambda^{\text{in}})} = 1$. We can write

$$\begin{pmatrix} \bar x \\ \bar\epsilon \\ \bar\mu \end{pmatrix} = \sum_{k\in V^{\text{out}}(L)} \bar\eta_k \begin{pmatrix} v_{\lambda^{\text{in}}} + \beta_k(L,\lambda^{\text{in}})(v^k - v_{\lambda^{\text{in}}}) \\ \beta_k(L,\lambda^{\text{in}})e^k \\ 0 \end{pmatrix} +$$
$$\sum_{j\in E\setminus E^\infty} \bar\kappa_j \begin{pmatrix} v_{\lambda^{\text{in}}} + \alpha_j(L,\lambda^{\text{in}})r^j \\ 0 \\ \alpha_j(L,\lambda^{\text{in}})e^j \end{pmatrix} + \sum_{j\in E^\infty} \bar\mu_j \begin{pmatrix} r^j \\ 0 \\ e^j \end{pmatrix},$$

where $\bar\kappa_j := \frac{\bar\mu_j}{\alpha_j(L,\lambda^{\text{in}})}$ for $j \in E\setminus E^\infty$ and $\bar\eta_k := \frac{\bar\epsilon_k}{\beta_k(L,\lambda^{\text{in}})}$ for $k \in V^{\text{out}}(L)$. Since $v_{\lambda^{\text{in}}}+ \alpha_j(L,\lambda^{\text{in}})r^j \notin \text{int}(L)$ for $j \in E\setminus E^\infty$, $v_{\lambda^{\text{in}}}+ \beta_k(L,\lambda^{\text{in}})(v^k - v_{\lambda^{\text{in}}}) \notin \text{int}(L)$ for $k \in V^{\text{out}}(L)$ and $(r^j,0,e^j)$ is in the recession cone of $R^l(L,P(\lambda^{\text{in}}))$ for $j\in E^\infty$, we have $(\bar x,\bar\epsilon,\bar\mu) \in R^l(L,P(\lambda^{\text{in}}))$.

(2) Now suppose $\sum_{j\in E} \frac{\bar\mu_j}{\alpha_j(L,\lambda^{\text{in}})} + \sum_{k\in V^{\text{out}}(L)} \frac{\bar\epsilon_k}{\beta_k(L,\lambda^{\text{in}})} > 1$ and $\sum_{k\in V^{\text{out}}(L)} \bar\epsilon_k = 1$. This implies $\bar x = \sum_{k\in V^{\text{out}}(L)} \bar\epsilon_k v^k + \sum_{j\in E} \bar\mu_j r^j$. Since $(r^j,0,e^j)$ is in the recession cone of $R^l(L,P,\lambda^{\text{in}})$ for $j \in E$, and since $v^k \notin \text{int}(L)$ for $k\in V^{\text{out}}(L)$, we have $(\bar x,\bar\epsilon,\bar\mu)\in R^l(L,P(\lambda^{\text{in}}))$.

(3) Next suppose $\sum_{j\in E} \frac{\bar\mu_j}{\alpha_j(L,\lambda^{\text{in}})} + \sum_{k\in V^{\text{out}}(L)} \frac{\bar\epsilon_k}{\beta_k(L,\lambda^{\text{in}})} > 1$ and $0 < \sum_{k\in V^{\text{out}}(L)} \bar\epsilon_k < 1$. Let $\bar\delta \in]0,1[$ be such that $\bar y := \bar\delta v_{\lambda^{\text{in}}} + (1-\bar\delta)\bar x = v_{\lambda^{\text{in}}}+ \sum_{k\in V^{\text{out}}(L)} (1-\bar\delta)\bar\epsilon_k(v^k - v_{\lambda^{\text{in}}}) + \sum_{j\in E}(1-\bar\delta)\bar\mu_j r^j$ satisfies $\sum_{j\in E} \frac{(1-\bar\delta)\bar\mu_j}{\alpha_j(L,\lambda^{\text{in}})} + \sum_{k\in V^{\text{out}}(L)} \frac{(1-\bar\delta)\bar\epsilon_k}{\beta_k(L,\lambda^{\text{in}})} = 1$. It follows from (1) that $(\bar y,(1-\bar\delta)\bar\epsilon,(1-\bar\delta)\bar\mu) \in R^l(L,P(\lambda^{\text{in}}))$. Let $d := \bar y - v_{\lambda^{\text{in}}}$, and consider the halfline $\{v_{\lambda^{\text{in}}} + \alpha d : \alpha \geq 0\}$. For $\alpha_{\bar y} := 1$, we have $v_{\lambda^{\text{in}}} + \alpha_{\bar y} d = \bar y$, and for $\alpha_{\bar x} := \frac{1}{1-\bar\delta}$, we have $v_{\lambda^{\text{in}}} + \alpha_{\bar x} d = \bar x$. Consider the point $\bar z := v_{\lambda^{\text{in}}} + \alpha_{\bar z} d$, where $\alpha_{\bar z} := \frac{1}{(1-\bar\delta)\sum_{k\in V^{\text{out}}(L)}\bar\epsilon_k}$. Since $\sum_{k\in V^{\text{out}}(L)} \bar\epsilon_k \in]0,1[$, we have $\alpha_{\bar y} < \alpha_{\bar x} < \alpha_{\bar z} < +\infty$. Hence $\bar x$ is a convex combination of $\bar y$ and $\bar z$. We may write $\bar z = v_{\lambda^{\text{in}}} + \alpha_{\bar z} d = v_{\lambda^{\text{in}}}+ \sum_{k\in V^{\text{out}}(L)} \alpha_{\bar z}(1-\bar\delta)\bar\epsilon_k(v^k - v_{\lambda^{\text{in}}}) + \sum_{j\in E} \alpha_{\bar z}(1-\bar\delta)\bar\mu_j r^j$. Observe that $\sum_{k\in V^{\text{out}}(L)} \alpha_{\bar z}(1-\bar\delta)\bar\epsilon_k = 1$. Hence we can write $\bar z = \sum_{k\in V^{\text{out}}(L)} \bar\eta_k v^k + \sum_{j\in E} \alpha_{\bar z}\bar\mu_j r^j$, where $\bar\eta_k := \alpha_{\bar z}(1-\bar\delta)\bar\epsilon_k$ for $k\in V^{\text{out}}(L)$ and $\sum_{k\in V^{\text{out}}(L)} \bar\eta_k = 1$. Since r^j is in the recession cone of $R(L,P(\lambda^{\text{in}}))$ for $j\in E$, and since $v^k \in R(L,P(\lambda^{\text{in}}))$ for $k\in V^{\text{out}}(L)$, we have $\bar z \in R(L,P(\lambda^{\text{in}}))$. Since $\bar x$ is a convex combination of $\bar y \in R(L,P(\lambda^{\text{in}}))$ and $\bar z \in R(L,P(\lambda^{\text{in}}))$, we have $\bar x \in R(L,P(\lambda^{\text{in}}))$.

(4) Finally suppose $\sum_{k\in V^{\text{out}}(L)} \bar\epsilon_k = 0$ and $\sum_{j\in E} \frac{\bar\mu_j}{\alpha_j(L,\lambda^{\text{in}})} > 1$. As in (3), let $\bar\delta \in]0,1[$ be s.t. $\bar y := \bar\delta v_{\lambda^{\text{in}}} + (1-\bar\delta)\bar x = v_{\lambda^{\text{in}}} + \sum_{j\in E}(1-\bar\delta)\bar\mu_j r^j$ satisfies $\sum_{j\in E} \frac{(1-\bar\delta)\bar\mu_j}{\alpha_j(L,\lambda)} = 1$. From (1) we have $(\bar y,0,(1-\bar\delta)\bar\mu) \in R^l(L,P(\lambda^{\text{in}}))$, and since $\bar y = \bar\delta v_{\lambda^{\text{in}}} + (1-\bar\delta)\bar x$, we have $\bar x = v_{\lambda^{\text{in}}}+ \bar\sigma(\bar y - v_{\lambda^{\text{in}}})$, where $\bar\sigma := \frac{1}{1-\bar\delta}$. Since $(\bar y, 0, \frac{\bar\mu}{\bar\sigma}) \in R^l(L,P(\lambda^{\text{in}}))$ satisfies $\sum_{j\in E} \frac{(1-\bar\delta)\bar\mu_j}{\alpha_j(L,\lambda^{\text{in}})} = 1$, (1) shows

$$\begin{pmatrix} \bar y \\ 0 \\ \frac{\bar\mu}{\bar\sigma} \end{pmatrix} = \sum_{j\in E\setminus E^\infty} \bar\kappa_j \begin{pmatrix} v_{\lambda^{\text{in}}} + \alpha_j(L,\lambda^{\text{in}})r^j \\ 0 \\ \alpha_j(L,\lambda^{\text{in}})e^j \end{pmatrix} + \sum_{j\in E^\infty} \bar\gamma_j \begin{pmatrix} r^j \\ 0 \\ e^j \end{pmatrix},$$

where $\sum_{j\in E\setminus E^\infty} \bar\kappa_j = 1$, $\bar\kappa_j \geq 0$ for $j \in E\setminus E^\infty$ and $\bar\gamma_j \geq 0$ for $j\in E^\infty$. We can now write

$$\begin{pmatrix} \bar x \\ 0 \\ \bar\mu \end{pmatrix} = \sum_{j\in E\setminus E^\infty} \bar\kappa_j \begin{pmatrix} v_{\lambda^{\text{in}}} + \bar\sigma\alpha_j(L,\lambda^{\text{in}})r^j \\ 0 \\ \bar\sigma\alpha_j(L,\lambda^{\text{in}})e^j \end{pmatrix} + \sum_{j\in E^\infty} \bar\sigma\bar\gamma_j \begin{pmatrix} r^j \\ 0 \\ e^j \end{pmatrix}.$$

□

4.3 The vertices of $R(L,P)$ The proof of Theorem 4.1 allows us to characterize the vertices of $R(L,P)$. Observe that in the proof of Theorem 4.1, every point in $R(L,P(\lambda^{\text{in}}))$ is expressed in terms of intersection points, vertices of P that are not in the interior of L and the extreme rays r^j of $R(L,P(\lambda^{\text{in}}))$ for $j\in E$. Hence the proof of Theorem 4.1 provides a characterization of the vertices of $R(L,P(\lambda^{\text{in}}))$.

Corollary 4.1 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$, and let $\lambda^{in} \in \Lambda^{in}(L)$. Define $E^\infty(\lambda^{in}) := \{j \in E : \alpha_j(L,\lambda^{in}) = +\infty\}$. A vertex of $R(L,P(\lambda^{in}))$ is of one of the following forms.*

(i) A vertex v^k of P, where $k \in V^{out}(L)$,

(ii) An intersection point $v_{\lambda^{in}} + \beta_k(L,\lambda^{in})(v^k - v_{\lambda^{in}})$, where $k \in V^{out}(L)$, or

(iii) An intersection point $v_{\lambda^{in}} + \alpha_j(L,\lambda^{in})r^j$, where $j \in E \setminus E^{\infty}(\lambda^{in})$.

By using the properties of $\alpha_j(L,\lambda^{\text{in}})$ and $\beta_k(L,\lambda^{\text{in}})$ for $\lambda^{\text{in}} \in \Lambda^{\text{in}}(L)$ given in Lemma 4.1 and Lemma 4.2, we can use Corollary 4.1 to characterize the vertices of $R(L,P)$. In the following, for simplicity let $\alpha_{i,j}(L) := \alpha_j(L,e^i)$ and $\beta_{i,k}(L) := \beta_k(L,e^i)$ for $i \in V^{\text{in}}(L)$, $j \in E$ and $k \in V^{\text{out}}(L)$. Also let $E^{\infty}(L) := \{j \in E : \alpha_{i,j}(L) = +\infty \text{ for some } i \in V^{\text{in}}(L)\}$ denote those extreme rays of P that are also rays of L.

Lemma 4.3 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$. Every vertex of $R(L,P)$ is of one of the following the forms.*

(i) A vertex v^k of P, where $k \in V^{out}(L)$,

(ii) An intersection point $v^i + \beta_{i,k}(L)(v^k - v^i)$, where $i \in V^{in}(L)$ and $k \in V^{out}(L)$, or

(iii) An intersection point $v^i + \alpha_{i,j}(L)r^j$, where $i \in V^{in}(L)$ and $j \in E \setminus E^{\infty}(L)$.

Proof. Let $\bar{x} \in R(L,P)$ be a vertex of $R(L,P)$, and let $(\bar{\lambda}^{\text{in}}, \bar{\epsilon}, \bar{\mu}) \in \mathbb{R}^{|V|+|E|}$ satisfy $\bar{x} = v_{\bar{\lambda}^{\text{in}}} + \sum_{k\in V^{\text{out}}(L)} \bar{\epsilon}_k(v^k - v_{\bar{\lambda}^{\text{in}}}) + \sum_{j\in E} \bar{\mu}_j r^j$, $\bar{\epsilon} \geq 0$, $\bar{\mu} \geq 0$, $\bar{\lambda}^{\text{in}} \in \Lambda^{\text{in}}(L)$ and $\sum_{k\in V^{\text{out}}(L)} \bar{\epsilon}_k \leq 1$. Now, we have $\bar{x} \in R(L,P(\bar{\lambda}^{\text{in}}))$, and since $R(L,P(\bar{\lambda}^{\text{in}})) \subseteq R(L,P)$, we must have that $\bar{x}$ is a vertex of $R(L,P(\bar{\lambda}^{\text{in}}))$. It follows that $\bar{x}$ is of one of the forms Corollary 4.1.(i)-(iii). If $\bar{x}$ is of the form $\bar{x} = v^k$ for some $k \in V^{\text{out}}(L)$, we are done. Furthermore, if $\bar{x} = v_{\bar{\lambda}^{\text{in}}} + \beta_k(L,\bar{\lambda}^{\text{in}})(v^k - v_{\bar{\lambda}^{\text{in}}})$ for some $k \in V^{\text{out}}(L)$, then Lemma 4.2 shows that either $\bar{x} = v^k$, or $\bar{x} = v^{\bar{i}} + \beta_{\bar{i},k}(L)(v^k - v^{\bar{i}})$ for some $\bar{i} \in V^{\text{out}}(L)$.

Finally consider the case when $\bar{x}$ is of the form $\bar{x} = v_{\bar{\lambda}^{\text{in}}} + \alpha_{\bar{j}}(L,\bar{\lambda}^{\text{in}})r^{\bar{j}}$ for some $\bar{j} \in E \setminus E^{\infty}(L)$. Since $\alpha_{\bar{j}}(L,\bar{\lambda}^{\text{in}})$ is concave in $\bar{\lambda}^{\text{in}}$, we have $\alpha_{\bar{j}}(L,\bar{\lambda}^{\text{in}}) \geq \sum_{i\in V^{\text{in}}(L)} \bar{\lambda}^{\text{in}}_i \alpha_{i,\bar{j}}(L)$. Let $\delta \geq 0$ satisfy $\alpha_{\bar{j}}(L,\bar{\lambda}^{\text{in}}) = \sum_{i\in V^{\text{in}}(L)} \bar{\lambda}^{\text{in}}_i(\alpha_{i,\bar{j}}(L) + \delta)$. We can now write $\bar{x}$ in the form $\bar{x} = v_{\bar{\lambda}^{\text{in}}} + \alpha_{\bar{j}}(L,\bar{\lambda}^{\text{in}})r^{\bar{j}} = \sum_{i\in V^{\text{in}}(L)} \bar{\lambda}^{\text{in}}_i(v_i + (\alpha_{i,\bar{j}}(L) + \delta)r^{\bar{j}})$. Since $v_i + (\alpha_{i,\bar{j}}(L) + \delta)r^{\bar{j}} \notin \text{int}(L)$ for all $i \in V^{\text{in}}(L)$, and $\bar{x}$ is a vertex of $R(L,P)$, we must have $\delta = 0$ and $\bar{\lambda}^{\text{in}}_{\bar{i}} = 1$ for some $\bar{i} \in V^{\text{in}}(L)$. $\square$

An important consequence of Lemma 4.3 is the following. For two mixed integer split polyhedra L^1 and L^2, if $V^{\text{in}}(L^1) = V^{\text{in}}(L^2)$, and if all the halflines $\{v^i + \alpha r^j : \alpha \geq 0\}$ and $\{v^i + \beta(v^k - v^i)\}$ for $i \in V^{\text{in}}(L^1) = V^{\text{in}}(L^2)$, $j \in E$ and $k \in V^{\text{out}}(L^1) = V^{\text{out}}(L^2)$ intersect the boundaries of L^1 and L^2 at the same points, then $R(L^1,P) = R(L^2,P)$. In other words, the relaxation of P_I obtained from L^1 is the same as the relaxation of P_I obtained from L^2.

Corollary 4.2 *Let L^1 and L^2 be mixed integer split polyhedra satisfying $V^{in}(L^1) = V^{in}(L^2) \neq \emptyset$. If $\alpha_{i,j}(L^1) = \alpha_{i,j}(L^2)$ and $\beta_{i,k}(L^1) = \beta_{i,k}(L^2)$ for all $i \in V^{in}(L^1) = V^{in}(L^2)$, $j \in E$ and $k \in V^{out}(L^1) = V^{out}(L^2)$, then $R(L^1,P) = R(L^2,P)$.*

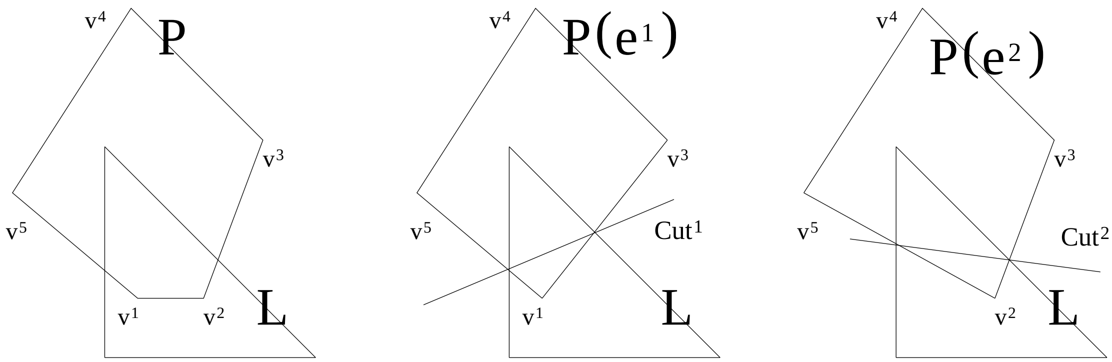
Another consequence of Lemma 4.3 is that it is possible to write $R(L,P)$ as the convex hull of the union of the polyhedra $R(L,P(e^i))$ for $i \in V^{\text{in}}(L)$.

Corollary 4.3 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$. We have*

$$R(L,P) = \text{conv}(\cup_{i\in V^{in}(L)} R(L,P(e^i))).$$

Proof. Lemma 4.3 shows that every vertex of $R(L,P)$ is a vertex of a set $R(L,P(e^i))$ for some $i \in V^{\text{in}}(L)$. Furthermore, the union of the vertices of the sets $R(L,P(e^i))$ over all $i \in V^{\text{in}}(L)$ is exactly the set of vertices of $R(L,P)$. Since the extreme rays of $R(L,P)$ and the sets $R(L,P(e^i))$ for $i \in V^{\text{in}}(L)$ are the same, namely the vectors $\{r^j\}_{j\in E}$, the result follows. $\square$

Figure 4 illustrates Corollary 4.3 on the example of Figure 1. The sets $P(e^1)$ and $P(e^2)$ corresponding to the two vertices v^1 and v^2 of P that are in the interior of L are shown in Figure 4.(b) and Figure 4.(c). Observe that the sets $R(L,P(e^1))$ and $R(L,P(e^2))$ are both described by adding exactly one cut to $P(e^1)$ and $P(e^2)$ respectively. Corollary 4.3 then shows that $R(L,P)$ can be obtained by taking the convex hull of the union of the sets $R(L,P(e^1))$ and $R(L,P(e^2))$.



(a) The polytope P and the split polyhedron L from Figure 1 (b) The set $P(e^1)$ constructed from v^1 (c) The set $P(e^2)$ constructed from v^2

Figure 4: Constructing $R(L, P)$ as the convex hull of the union of polyhedra

4.4 Polyhedrality of the w^{th} split closure We now use Theorem 3.1 to prove that the w^{th} split closure of P is a polyhedron. Let $L \in \mathcal{L}^w$ be an arbitrary mixed integer split polyhedron, where $w > 0$, and let $\delta^T x \geq \delta_0$ be a valid inequality for $R(L, P)$ with integral coefficients which is not valid for P. To use Theorem 3.1, we consider potential intersection points between the hyperplane $\delta^T x = \delta_0$ and halflines of the form $\{v^i + \alpha r^j : \alpha \geq 0\}$, and of the form $\{v^i + \beta(v^k - v^i) : \beta \geq 0\}$, where $i \in V^{\text{in}}(L)$, $j \in E$ and $k \in V^{\text{out}}(L)$. The properties we derive of these intersection points do not depend on the particular halfline, so we only consider the halfline $\{v^1 + \alpha r^1 : \alpha \geq 0\}$. We will show that the rationality of v^1 and r^1 can be used to limit the number of possible intersection points. This then allows us to conclude that the w^{th} split closure is a polyhedron. We first give a representation of $\alpha'_{1,1}(\delta, \delta_0)$ for a given valid inequality $\delta^T x \geq \delta_0$ for $v^1 + \alpha_{1,1}(L) r^1$.

Lemma 4.4 *(Lemma 5 in [1]). Let $L \in \mathcal{L}^w$ be a mixed integer split polyhedron with max-facet-width at most $w > 0$. Suppose $v^1 \in \text{int}(L)$ and $\alpha_{1,1}(L) < +\infty$, and also suppose $\delta^T x \geq \delta_0$ is a non-negative cut for $\{v^1 + \alpha r^1 : \alpha \geq 0\}$ with integral coefficients that is valid for $v^1 + \alpha_{1,1}(L) r^1$.*

(i) $0 < \alpha'_{1,1}(\delta, \delta_0) \leq \alpha_{1,1}(L) < w$, and

(ii) $\alpha'_{1,1}(\delta, \delta_0) = \frac{s(\delta,\delta_0)}{g t(\delta,\delta_0)}$, where g, $s(\delta, \delta_0)$,$t(\delta, \delta_0) > 0$ are integers satisfying $s(\delta, \delta_0) < gw$.

(Note that the integer g is independent of both L and $\delta^T x \geq \delta_0$).

Proof. We may write $L = \{x \in \mathbb{R}^n : (\pi^k)^T x \geq \pi_0^k \text{ for } k \in N_f\}$, where $N_f := \{1, 2, \ldots, n_f\}$, n_f denotes the number of facets of L and $(\pi^k, \pi_0^k) \in \mathbb{Z}^{n+1}$ for $k \in N_f$. Since $v^1 \in \text{int}(L)$, we have $(\pi^k)^T v^1 < \pi_0^k$ for all $k \in N_f$, and therefore $\alpha_{1,1}(L) = \frac{\pi_0^{\bar{k}} - (\pi^{\bar{k}})^T v^1}{(\pi^k)^T r^1}$ for some $\bar{k} \in N_f$. Since L has max-facet-width at most w and $v^1 \in \text{int}(L)$, we have $0 < \pi_0^{\bar{k}} - (\pi^{\bar{k}})^T v^1 < w$. Hence, since $(\pi^{\bar{k}})^T r^1$ is integer, we have $(\pi^{\bar{k}})^T r^1 \geq 1$, and therefore $\alpha_{1,1}(L) < w$. Furthermore, since $\delta^T x \geq \delta_0$ is a non-negative cut for the set $\{v^1 + \alpha r^1 : \alpha \geq 0\}$ that is valid for $v^1 + \alpha_{1,1}(L) r^1$, we have $\alpha'_{1,1}(\delta, \delta_0) \leq \alpha_{1,1}(L)$.

Recall that we assumed $v^1 \in \mathbb{Q}^n$ and $r^1 \in \mathbb{Z}^n$. We can therefore write $v^1 = (\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n})$, where $p_k \in \mathbb{Z}$ and $q_k \in \mathbb{N}$ for $k = 1, 2, \ldots, n$. Define the integers $g := \Pi_{k=1}^n q_k$, $d_m := \Pi_{k=1, k \neq m}^n q_k$ for $m \in \{1, 2, \ldots, n\}$, $s(\delta, \delta_0) := g\delta_0 - \sum_{m=1}^n d_m p_m \delta_m$ and $t(\delta, \delta_0) := \delta^T r^1$. Observe that $\frac{s(\delta,\delta_0)}{g} = \delta_0 - \delta^T v^1$. With these choices, (ii) is satisfied. □

By using the above lemma, we can now bound the number of possible intersection points with a halfline of the form $\{v^1 + \alpha r^1 : \alpha \geq \alpha^*\}$ for some $\alpha^* > 0$.

Lemma 4.5 *(Lemma 6 in [1]). Let $\alpha^* > 0$ and $w > 0$. Also let $\{(\delta^l)^T x \geq \delta_0^l\}_{l \in I}$ be a set of non-negative cuts for $\{v^1 + \alpha r^1 : \alpha \geq 0\}$ with integral coefficients that are all valid for a point $v^1 + \alpha_{1,1}(L) r^1$ for some $L \in \mathcal{L}^w$. The set $\{\alpha'_{1,1}(\delta^l, \delta_0^l) : l \in I$ and $\alpha'_{1,1}(\delta^l, \delta_0^l) \geq \alpha^*\}$ is finite.*

Proof. Let $l \in I$ satisfy $\alpha^* \leq \alpha'_{1,1}(\delta^l, \delta^l_0) \leq +\infty$. We may assume α^* is of the form $\alpha^* = \frac{s^*}{gt^*}$ for some integers $s^*, t^* > 0$ satisfying $0 < s^* < gw$.

Let $s(\delta^l, \delta^l_0)$ and $t(\delta^l, \delta^l_0)$ be as in Lemma 4.4. Hence we have $\alpha'_{1,1}(\delta^l, \delta^l_0) = \frac{s(\delta^l,\delta^l_0)}{gt(\delta^l,\delta^l_0)}$. This implies $s(\delta^l, \delta^l_0) \in \{1, 2, \dots, (gw-1)\}$, so there is only a finite number of possible values for $s(\delta^l, \delta^l_0)$. Finally, Lemma 4.4.(i) and $\alpha'_{1,1}(\delta^l, \delta^l_0) \geq \alpha^*$ gives $\frac{s^*}{gt^*} \leq \frac{s(\delta^l,\delta^l_0)}{gt(\delta,\delta_0)} < w$, and therefore $\frac{s(\delta^l,\delta^l_0)}{gw} < t(\delta^l, \delta^l_0) \leq \frac{s(\delta^l,\delta^l_0)t^*}{s^*}$. Hence, for a fixed value $s(\delta^l, \delta^l_0) \in \{1, 2, \dots, (gw-1)\}$, there is only a finite number of possible values for $t(\delta^l, \delta^l_0)$. □

By using Lemma 4.5, we can now conclude that the w^{th} split closure is a polyhedron.

Theorem 4.2 *Let $\bar{\mathcal{L}} \subseteq \mathcal{L}^w$ be any family of mixed integer split polyhedra that have max-facet-width at most $w > 0$. The set $\cap_{L \in} R(L, P)$ is a polyhedron.*

Proof. Let $\{(\delta^l)^T x \geq \delta^l_0\}_{l \in I}$ denote the family of all cuts for P that are valid and facet defining for $R(L, P)$ for some $L \in \bar{\mathcal{L}}$. As discussed in Sect. 3.3, we can partition the cuts in I into a finite number of subsets $I^c(S) \subseteq I$ according to which set $S \subseteq V$ of vertices they cut off. Lemma 4.5 shows that Assumption 3.1 of Sect. 3.3 is satisfied by each set $\{(\delta^l)^T x \geq \delta^l_0\}_{l \in I^c(S)}$. □

5. Finite split polyhedron proofs Mixed integer split polyhedra can be used to design finite cutting plane proofs for the validity of an inequality for P_I as follows. Let $\delta^T x \geq \delta_0$ be a valid inequality for P_I. Observe that, if $\delta^T x \geq \delta_0$ is valid for $R(L, P)$ for some mixed integer split polyhedron L, then L provides a finite cutting plane proof of validity of $\delta^T x \geq \delta_0$ for P_I. More generally, a family $\mathcal{S}$ of mixed integer split polyhedra gives an approximation of P_I of the form

$$\mathrm{Cl}(\mathcal{S}, P) := \bigcap_{L \in \mathcal{S}} R(L, P).$$

The set $\mathrm{Cl}(\mathcal{S}, P)$ gives the closure wrt. the family $\mathcal{S}$. Improved approximations of P_I can be obtained by iteratively computing closures $P^1(\mathcal{S}, P), P^2(\mathcal{S}, P), P^3(\mathcal{S}, P) \dots$, where $P^0(\mathcal{S}, P) = P$, $P^1(\mathcal{S}, P) = \mathrm{Cl}(\mathcal{S}, P^0(\mathcal{S}, P))$, $P^2(\mathcal{S}, P) = \mathrm{Cl}(\mathcal{S}, P^1(\mathcal{S}, P))$ etc. A *finite split polyhedron proof* of validity of $\delta^T x \geq \delta_0$ for P_I is a finite family $\mathcal{S}$ of mixed integer split polyhedra such that $\delta^T x \geq \delta_0$ is valid for $P^k(\mathcal{S}, P)$ for some $k < \infty$, and a finite cutting plane proof is given from a finite split polyhedron proof by the valid inequalities for the polyhedron $P^k(\mathcal{S}, P)$.

A measure of the complexity of a finite split polyhedron proof $\mathcal{S}$ is the max-facet-width of the mixed integer split polyhedron $L \in \mathcal{S}$ with the largest max-facet-width. We call this number the *width size* of a split polyhedron proof. A measure of the complexity of a valid inequality $\delta^T x \geq \delta_0$ for P_I is then the smallest number w for which there exists a finite split polyhedron proof of validity of $\delta^T x \geq \delta_0$ for P_I of width size w. This number is called the width size of $\delta^T x \geq \delta_0$, and it is denoted width-size(δ, δ_0). Finally, since validity of every facet defining inequality for $\mathrm{conv}(P_I)$ must be proved to generate $\mathrm{conv}(P_I)$, the largest of the numbers width-size(δ, δ_0) over all facet defining inequalities $\delta^T x \geq \delta_0$ for $\mathrm{conv}(P_I)$ gives a measure of the complexity of P_I. We call this number the width size of P_I, and it is denoted width-size(P_I). We give an example to show that width-size(P_I) can be as large as the number of integer constrained variables at the end of this section.

We now characterize exactly which max-facet-width is necessary to prove validity of an inequality $\delta^T x \geq \delta_0$ for P_I with a finite split polyhedron proof, *i.e.*, we characterize the number width-size(δ, δ_0). We will partition the inequality $\delta^T x \geq \delta_0$ into its integer part and its continuous part. Throughout the remainder of this section, $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ denotes an arbitrary valid inequality for P_I, where $\delta^x \in \mathbb{Q}^p$, $\delta^y \in \mathbb{Q}^q$ and $\delta_0 \in \mathbb{Q}$. We assume $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ is tight at a mixed integer point of P_I.

It is possible to prove validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for $\mathrm{conv}(P_I)$ by solving the mixed integer linear problem (MIP)

$$\begin{aligned} &\min\, (\delta^x)^T x + (\delta^y)^T y \\ &s.t. \\ &\qquad (x, y) \in \ P_I. \end{aligned}$$

The following notation is used. The point $(x^*, y^*) \in P_I$ denotes an optimal solution to MIP, and $(x^{\mathrm{lp}}, y^{\mathrm{lp}}) \in P$ denotes an optimal solution to the linear relaxation of MIP. We assume $\delta_0 = (\delta^x)^T x^* + (\delta^y)^T y^*$ and $(\delta^x)^T x^{\mathrm{lp}} + (\delta^y)^T y^{\mathrm{lp}} < \delta_0$. From the inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$, we can create the following subsets of P and P_I

$$\begin{aligned} P(\delta, \delta_0) &:= \{(x, y) \in P : (\delta^x)^T x + (\delta^y)^T y \leq \delta_0\} \text{ and} \\ P_I(\delta, \delta_0) &:= \{(x, y) \in P(\delta, \delta_0) : x \in \mathbb{Z}^p\}. \end{aligned}$$

To prove validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for $\mathrm{conv}(P_I)$, we consider the following projections of $P(\delta, \delta_0)$ and $P_I(\delta, \delta_0)$ onto the space of the integer constrained x variables

$$\begin{aligned} P^x(\delta, \delta_0) &:= \{x \in \mathbb{R}^p : \exists y \in \mathbb{R}^q \text{ such that } (x, y) \in P(\delta, \delta_0)\} \text{ and} \\ P_I^x(\delta, \delta_0) &:= P^x(\delta, \delta_0) \cap \mathbb{Z}^p. \end{aligned}$$

The validity proofs we derive for $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ are based on the following important property.

Lemma 5.1 *The polyhedron $P^x(\delta, \delta_0)$ is lattice point free.*

Proof. The relative interior of $P^x(\delta, \delta_0)$ is given by

$$\mathrm{ri}(P^x(\delta, \delta_0)) = \{x \in \mathbb{R}^p : \exists y \in \mathbb{R}^q \text{ such that } (x, y) \in \mathrm{ri}(P) \text{ and } (\delta^x)^T x + (\delta^y)^T y < \delta_0\}.$$

Since δ_0 is the optimal objective value of MIP, $\mathrm{ri}(P^x(\delta, \delta_0))$ does not contain lattice points. □

It is well known that split polyhedra with max-facet-width equal to one are sufficient to generate the integer hull of a pure integer set. It follows from this result that there exists a finite number of split polyhedra with max-facet-width equal to one such that a polyhedron $\overline{P}$ can be obtained in a finite number of iterations that satisfies $\overline{P}^x(\delta, \delta_0) = \mathrm{conv}(\overline{P}_I^x(\delta, \delta_0))$. Hence, since the purpose in this section is to provide finite split polyhedron proofs, we can assume $P^x(\delta, \delta_0) = \mathrm{conv}(P_I^x(\delta, \delta_0))$ in the remainder of this section.

The split polyhedra that are needed to prove validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for P_I depend on the facial structure of $P^x(\delta, \delta_0)$. To obtain a description of the faces of $P^x(\delta, \delta_0)$, we need the following reformulation of $P^x(\delta, \delta_0)$.

Lemma 5.2 *Assume $P^x(\delta, \delta_0) = \mathrm{conv}(P_I^x(\delta, \delta_0))$. For every $x \in P^x(\delta, \delta_0)$, there exists $y \in \mathbb{R}^q$ such that $(x, y) \in P$ and $(\delta^x)^T x + (\delta^y)^T y = \delta_0$. Hence*

$$P^x(\delta, \delta_0) = \{x \in \mathbb{R}^p : \textit{ there exists } y \in \mathbb{R}^q \textit{ s.t. } (x, y) \in P \textit{ and } (\delta^x)^T x + (\delta^y)^T y = \delta_0\}.$$

Proof. First suppose $\bar{x} \in P^x(\delta, \delta_0)$ is integer. By definition of $P^x(\delta, \delta_0)$, there exists $\bar{y} \in \mathbb{R}^q$ such that $(\bar{x}, \bar{y}) \in P$ and $(\delta^x)^T \bar{x} + (\delta^y)^T \bar{y} \leq \delta_0$. We can not have $(\delta^x)^T \bar{x} + (\delta^y)^T \bar{y} < \delta_0$, since δ_0 is the optimal objective of MIP. Hence $(\bar{x}, \bar{y}) \in P$ and $(\delta^x)^T \bar{x} + (\delta^y)^T \bar{y} = \delta_0$.

Now suppose $x^r \in \mathbb{Q}^p$ is a ray of $P^x(\delta, \delta_0)$. We claim that for every $\mu \geq 0$ and $\bar{x} \in P_I^x(\delta, \delta_0)$, there exists $\bar{y} \in \mathbb{R}^q$ such that $(\bar{x} + \mu x^r, \bar{y}) \in P$ and $(\delta^x)^T (\bar{x} + \mu x^r) + (\delta^y)^T \bar{y} = \delta_0$. Indeed, let $\mu \geq 0$ and $\bar{x} \in P_I^x(\delta, \delta_0)$ be arbitrary. We can choose a non-negative integer $\mu^I \geq \mu$ such that $\bar{x} + \mu^I x^r$ is integer. We therefore have that there exists $y^1 \in \mathbb{R}^q$ such that $(\bar{x} + \mu^I x^r, y^1) \in P$ and $(\delta^x)^T (\bar{x} + \mu^I x^r) + (\delta^y)^T y^1 = \delta_0$. Since $\bar{x} \in P_I^x(\delta, \delta_0)$, we also have that there exists $y^2 \in \mathbb{R}^q$ such that $(\bar{x}, y^2) \in P$ and $(\delta^x)^T \bar{x} + (\delta^y)^T y^2 = \delta_0$. By choosing $\lambda := \frac{\mu}{\mu_I}$ and $\bar{y} := \lambda y^1 + (1 - \lambda) y^2$, we have $(\bar{x} + \mu x^r, \bar{y}) = \lambda(\bar{x} + \mu^I x^r, y^1) + (1 - \lambda)(\bar{x}, y^2)$, and therefore $(\bar{x} + \mu x^r, \bar{y}) \in P$. In addition we have that $(\delta^x)^T (\bar{x} + \mu x^r) + (\delta^y)^T \bar{y} = \delta_0$.

Finally let $\bar{x} \in P^x(\delta, \delta_0)$ be arbitrary. We may write $\bar{x} = \sum_{i=1}^k \lambda_i x^i + d = \sum_{i=1}^k \lambda_i (x^i + d)$, where $\{x^i\}_{i=1}^k$ are the vertices of $P^x(\delta, \delta_0)$, $d \in \mathbb{Q}^p$ is a non-negative combination of the extreme rays of $P^x(\delta, \delta_0)$, $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$. From what was shown above, we have that for every $i \in \{1, 2, \ldots, k\}$, there exists $y^i \in \mathbb{R}^q$ such that $(x^i + d, y^i) \in P$ and $(\delta^x)^T (x^i + d) + (\delta^y)^T y^i = \delta_0$. By letting $\bar{y} := \sum_{i=1}^k \lambda_i y^i$, we have that $(\bar{x}, \bar{y}) \in P$ and $(\delta^x)^T \bar{x} + (\delta^y)^T \bar{y} = \delta_0$. □

The faces of $P^x(\alpha,\beta)$ can now be characterized. Let $P = \{(x,y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + Dy \leq b\}$ be an outer description of P, where $A \in \mathbb{Q}^{m\times p}$, $D \in \mathbb{Q}^{m\times q}$ and $b \in \mathbb{Q}^m$, and let $M := \{1, 2, \ldots, m\}$. Lemma 5.2 shows that $P^x(\delta,\delta_0)$ can be written in the form

$$\begin{aligned} P^x(\delta,\delta_0) = \{x \in \mathbb{R}^p : \quad & a_{i.}^T x + d_{i.}^T y = b_i, i \in M^=, \\ & a_{i.}^T x + d_{i.}^T y \leq b_i, i \in M \setminus M^=, \\ & (\delta^x)^T x + (\delta^y)^T y = \delta_0\}, \end{aligned}$$

where $M^= \subseteq M$ denotes those constrains $i \in M$ for which $a_{i.}^T x + d_{i.} y = b_i$ for all $(x,y) \in P(\delta,\delta_0)$ that satisfy $(\delta^x)^T x + (\delta^y)^T y = \delta_0$. Also, for every $i \in M \setminus M^=$, there exists $(x,y) \in P(\delta,\delta_0)$ that satisfies $(\delta^x)^T x + (\delta^y)^T y = \delta_0$ and $a_{i.}^T x + d_{i.} y < b_i$.

A non-empty face F of $P^x(\delta,\delta_0)$ can be characterized by a set $M^F \subseteq M$ of inequalities that satisfies $M^= \subseteq M^F$. Every face F of $P^x(\delta,\delta_0)$ can be written in the form

$$\begin{aligned} F = \{x \in \mathbb{R}^p : \quad & a_{i.}^T x + d_{i.}^T y = b_i, i \in M^F, \\ & a_{i.}^T x + d_{i.}^T y \leq b_i, i \in M \setminus M^F, \\ & (\delta^x)^T x + (\delta^y)^T y = \delta_0\}. \end{aligned}$$

Consider an arbitrary proper face F of $P^x(\delta,\delta_0)$. In order for $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ to be valid for P_I, $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ must be valid for all $(x,y) \in P$ such that $x \in F$. The following lemma shows that F is of exactly one of two types depending on the coefficient vectors on the continous variables in the tight constraints.

Lemma 5.3 *(A characterization of the faces of $P^x(\delta,\delta_0)$)*
Assume $P^x(\delta,\delta_0) = \mathrm{conv}(P_I^x(\delta,\delta_0))$. Let F be a face of $P^x(\delta,\delta_0)$.

(i) If $\delta^y \notin \mathrm{span}(\{d_{i.}\}_{i\in M^F})$:

(a) F is lattice point free.

(b) For every $x \in \mathrm{ri}(F)$, there exists $y \in \mathbb{R}^q$ s.t. $(x,y) \in P$ and $(\delta^x)^T x + (\delta^y)^T y < \delta_0$.

(ii) If $\delta^y \in \mathrm{span}(\{d_{i.}\}_{i\in M^F})$:
The inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ holds for all $(x,y) \in P$ satisfying $x \in \mathrm{ri}(F)$.

Proof. (i) Suppose $\delta^y \notin \mathrm{span}(\{d_{i.}\}_{i\in M^F})$, and let $\bar{x} \in \mathrm{ri}(F)$ be arbitrary. This implies there exists $\bar{y} \in \mathbb{R}^q$ such that $a_{i.}^T \bar{x} + d_{i.}^T \bar{y} < b_i$ for all $i \in M \setminus M^F$. Since $\delta^y \notin \mathrm{span}(\{d_{i.}\}_{i\in M^F})$, the linear program $\min\{(\delta^y)^T r : d_{i.}^T r = 0, \forall i \in M^F\}$ is unbounded. Choose $\bar{r} \in \mathbb{R}^q$ such that $(\delta^y)^T \bar{r} < 0$ and $d_{i.}^T \bar{r} = 0$ for all $i \in M^F$. We have that $(\delta^x)^T \bar{x} + (\delta^y)^T(\bar{y} + \mu\bar{r}) < (\delta^x)^T \bar{x} + (\delta^y)^T \bar{y} = \delta_0$ for every $\mu > 0$. Furthermore, since $(\bar{x},\bar{y})$ satifies $a_{i.}^T \bar{x} + d_{i.}^T \bar{y} < b_i$ for all $i \in M \setminus M^F$, there exists $\bar{\mu} > 0$ such that $(\bar{x}, \bar{y} + \bar{\mu}\bar{r}) \in P$ and $(\delta^x)^T \bar{x} + (\delta^y)^T(\bar{y} + \bar{\mu}\bar{r}) < \delta_0$. We can not have $\bar{x}$ integer, since this would contradict that δ_0 is the optimal objective of MIP.

(ii) Let $(\bar{x},\bar{y}) \in P$ satisfy $\bar{x} \in \mathrm{ri}(F)$, and suppose $\delta^y \in \mathrm{span}(\{d_{i.}\}_{i\in M^F})$. If $(\delta^x)^T \bar{x} + (\delta^y)^T \bar{y} \geq \delta_0$, we are done, so suppose for a contradiction that $(\delta^x)^T \bar{x} + (\delta^y)^T \bar{y} < \delta_0$. Since $\bar{x} \in \mathrm{ri}(F)$, there exists $\tilde{y} \in \mathbb{R}^q$ such that $(\bar{x},\tilde{y}) \in P$, $(\delta^x)^T \bar{x} + (\delta^y)^T \tilde{y} = \delta_0$ and $a_{i.}^T \bar{x} + d_{i.}^T \tilde{y} < b_i$ for all $i \in M \setminus M^F$. Consider the vector $\bar{r} := \bar{y} - \tilde{y}$. We have $d_{i.}^T \bar{r} = 0$ for all $i \in M^F$ and $(\delta^y)^T \bar{r} < 0$. However, this contradicts $\delta^y \in \mathrm{span}(\{d_{i.}\}_{i\in M^F})$. □

We can now identify the mixed integer split polyhedra that are needed to provide a finite split polyhedron proof of validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for P_I. Let $\mathcal{F}$ denote the finite set of all faces of $P^x(\delta,\delta_0)$, and let $\mathcal{F}^V := \{F \in \mathcal{F} : \exists (x,y) \in P \text{ s.t. } x \in F \text{ and } (\delta^x)^T x + (\delta^y)^T y < \delta_0\}$ denote those faces $F \in \mathcal{F}$ for which there exists $(x,y) \in P$ such that $x \in F$ and (x,y) violates the inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$. A face $F \in \mathcal{F}^V$ is called a *violated face*. Lemma 5.3.(i) shows that every violated face is lattice point free. A mixed integer split polyhedron $L \subseteq \mathbb{R}^n$ that satisfies $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for every $(x,y) \in R(L,P)$ such that $x \in F$ is said to *prove validity of* $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ *on* F. Given a violated face $F \in \mathcal{F}^V$, the following lemma gives a class of split polyhedra that can prove validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ on F.

Lemma 5.4 *(Split polyhedra for proving validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ on a face of $P^x(\delta, \delta_0)$)* *Assume $P^x(\delta,\delta_0) = \mathrm{conv}(P_I^x(\delta,\delta_0))$. Let $F \in \mathcal{F}^V$ be a violated face of $P^x(\delta,\delta_0)$, and suppose $G \notin \mathcal{F}^V$ for every proper face G of F. Every mixed integer split polyhedron $L \subseteq \mathbb{R}^n$ that satisfies $\mathrm{ri}(F) \subseteq \mathrm{int}(L)$ proves validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ on F.*

Proof. Let L be a mixed integer split polyhedron that satisfies $\mathrm{ri}(F) \subseteq \mathrm{int}(L)$, and let $(\bar{x}, \bar{y}) \in P$ satisfy $\bar{x} \in F$ and $\bar{x} \notin \mathrm{int}(L)$. Since $\mathrm{ri}(F) \subseteq \mathrm{int}(L)$, it follows that $\bar{x} \notin \mathrm{ri}(F)$. Since $\bar{x} \in F \setminus \mathrm{ri}(F)$, $\bar{x}$ must be on some proper face G of F. Since $G \notin \mathcal{F}^V$, we have $(\delta^x)^T\bar{x} + (\delta^y)^T\bar{y} \geq \delta_0$. Since $R(L,F) = \mathrm{conv}(\{(x,y) \in F : x \notin \mathrm{int}(L)\})$, the result follows. □

By iteratively considering the finite number $|\mathcal{F}^V|$ of violated faces of $P^x(\delta,\delta_0)$, we obtain a finite split polyhedron proof for the validity of the inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for P_I.

Corollary 5.1 *(Upper bound on the width size of the inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$)* *There exists a split polyhedron proof for the validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for P_I of width size*

$$\max\{\textit{width-size}(F) : F \in \mathcal{F}^V\}.$$

We can now prove the main theorem of this section.

Theorem 5.1 *(A formula for the width size of the inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$)* *Let width-size$(\delta,\delta_0)$ denote the smallest number w for which there exists a finite split polyhedron proof of validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for P_I of width size w. Then*

$$\textit{width-size}(\delta,\delta_0) = \max\{\textit{width-size}(F) : F \in \mathcal{F}^V\}.$$

Proof. Let L be a mixed integer split polyhedron of smaller width size than $\max\{\text{width-size}(F) : F \in \mathcal{F}^V\}$. This implies there exists $F \in \mathcal{F}^V$ and $x' \in \mathrm{ri}(F)$ such that $x' \notin \mathrm{int}(L)$. Furthermore, since $x' \in \mathrm{ri}(F)$, it follows from Lemma 5.3.(i) that there exists $y' \in \mathbb{R}^q$ such that $(x',y') \in P$ and $(\delta^x)^T x' + (\delta^y)^T y' < \delta_0$. We now have $(x',y') \in R(L,P)$ and $(\delta^x)^T x' + (\delta^y)^T y' < \delta_0$. □

Example 5.1 *Consider the mixed integer linear program (MILP)*

$$\begin{aligned} &\max y \\ &s.t. \\ -x_i + y &\leq 0, \qquad \textit{for } i = 1,2,\ldots,p, && (17)\\ \sum_{i=1}^{p} x_i + y &\leq p, && (18)\\ y &\geq 0, && (19)\\ x_i &\textit{ integer} \quad \textit{for } i = 1,2,\ldots,p. && (20) \end{aligned}$$

The optimal solutions to MILP are of the form $(x^*,y^*) = (x^*,0)$ with $x^* \in S^p \cap \mathbb{Z}^p$, where $S^p := \{x \in \mathbb{R}^p : x \geq 0$ and $\sum_{i=1}^p x_i \leq p\}$. The unique optimal solution to the LP relaxation of MILP is given by $x_i^{lp} = \frac{p}{p+1}$ for $i = 1,2,\ldots,p$ and $y^{lp} = \frac{p}{p+1}$. Hence the only missing inequality to describe $\mathrm{conv}(P_I)$ is the inequality $y \leq 0$. We have $\delta^x = 0$, $\delta^y = -1$ and $\delta_0 = 0$.

Observe that any proper face G of $P^x(\delta,\delta_0)$ contains mixed integer points in their relative interior. It follows that the inequality $y \leq 0$ is valid for every $(x,y) \in P$ such that x belongs to a proper face of $P^x(\delta,\delta_0)$. Hence the only interesting face of $P^x(\delta,\delta_0)$ to consider is the improper face $F := P^x(\delta,\delta_0) = S^p$. The only mixed integer split polyhedron L that satisfies $\mathrm{ri}(F) \subseteq \mathrm{int}(L)$ is the split polyhedron $L = S^p$, and this mixed integer split polyhedron has max-facet-width p. It follows from Theorem 5.1 that no cutting plane algorithm that only uses mixed integer split polyhedra of max-facet-width smaller than p can solve MILP in a finite number of steps.

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